Lecture #21

Last time: - recalled basics of simple fin. dim. Lie algebras + Chevalley-Serre realization
- introduced contragredient Lie algebras $\mathfrak{g}(A)$
- presented Chevalley-Serre realization for $\mathfrak{g}(A)$ when A - generalized Cartan

Today: Relate this to affine Lie algebras from Lecture 2.

Def 1: A generalized indecomposable Cartan matrix A is called *affine* if it's diagonalizable and can be done so that $D=\mathrm{diag}(d_i)$ with $d_i \in \mathbb{R}_{>0}$ satisfies $D\cdot A=(D\cdot A)^T$ and defines a positive semidefinite form (but not positive definite)

The main result from today shows that $\hat{\mathfrak{g}}$ are such for simple $\mathfrak{g}$.

Input: • $\mathfrak{g}$ - simple Lie algebra, A - Cartan matrix of $\mathfrak{g}$, so that $a_{ij}=\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$

• $(\,,\,)$ - invariant symmetric form on $\mathfrak{g}$

Know: $(\,,\,):\mathfrak{h}\times\mathfrak{h}\to\mathbb{C}$ is nondegenerate $\Rightarrow$ allows to identify $\mathfrak{h}\simeq\mathfrak{h}^*$ and also pull $(\,,\,)$ to form on $\mathfrak{h}^*$

Normalization of $(\,,\,)$ is as before: $(\theta,\theta)=2$, where θ = highest root, i.e. the one with maximal sum of coeff-s in $\theta=\sum k_i \alpha_i$ ($k_i \in \mathbb{Z}_{\geq 0}$, α_i simple)

Note: $e_\alpha, f_\alpha, h_\alpha$ satisfy sl_2-rel-s imply $\underbrace{[h_\alpha,e_\alpha]}_{2e_\alpha}=\alpha(h_\alpha)\cdot e_\alpha \Rightarrow$

$\Rightarrow \alpha(h_\alpha)=2$, which implies that under above identification

$$h_\alpha\in\mathfrak{h} \longleftrightarrow \alpha^\vee=\frac{2}{(\alpha,\alpha)}\alpha\in\mathfrak{h}^*$$

in particular:

$$a_{ij}=\alpha_j(h_{\alpha_i})=\frac{2}{(\alpha_i,\alpha_i)}(\alpha_i,\alpha_j)$$

Theorem 1: $\hat{\mathfrak{g}}$ is an affine Kac-Moody algebra with affine Cartan matrix

$$\hat{A}=\begin{pmatrix} 2 & a_{0i} \\ a_{i0} & A \end{pmatrix}$$

with a_{0i}, a_{i0} expressed below

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This result is the main reason why affine Kac-Moody algebras are so well-studied

► Let $\{e_i, f_i, h_i\}_{i=1}^n$ be the Chevalley generators of $\mathfrak{g}$, as discussed last time.

• Step 1: Generators

Let $I = \{1, \dots, n\}$ and $\hat{I} = I \cup \{0\}$. For $i \in \hat{I} \setminus \{0\}$, we define e_i, f_i, h_i as the corresponding el-s of $\mathfrak{g} = \mathfrak{g} \cdot t^0 \subseteq \hat{\mathfrak{g}}$. Let θ = highest root of $\mathfrak{g}$. Then set: $\boxed{e_0 := f_\theta \cdot t,\ f_0 := e_\theta \cdot t^{-1},\ \ h_0 := K - h_\theta}$

Remark: For $\mathfrak{g} = \mathfrak{sl}_n$, we have $\theta = \alpha_1 + \dots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n \Rightarrow e_\theta = E_{1n},\ f_\theta = E_{n1}$
$\Rightarrow e_0 = E_{n1} t,\ f_0 = E_{1n} t^{-1},\ h_0 = K - E_{11} + E_{nn}$ which we already encountered as $\mathfrak{sl}_2^{(0)}$ generators in the end of Lecture 17

We first claim that these $\{e_i, f_i, h_i\}_{i \in \hat{I}}$ generate all $\hat{\mathfrak{g}}$. Indeed, taking $i \in I \subseteq \hat{I}$ we can generate all $\mathfrak{g} \cdot t^0 \subseteq \hat{\mathfrak{g}}$. As $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}t^n \subseteq \hat{\mathfrak{g}}$ is isomorphic to $ad: \mathfrak{g} \curvearrowright \mathfrak{g}$ and $\mathfrak{g}$-simple $\Rightarrow$ any $x \cdot t^n$ with $x \in \mathfrak{g} \setminus 0$, $n \in \mathbb{Z}$ would generate all $\mathfrak{g} \cdot t^n$ under this action. Thus, we can generate all $\mathfrak{g}t, \mathfrak{g}t^{-1}$. But these respectively generate all $\mathfrak{g}t^n, \mathfrak{g}t^{-n}$ $\forall n > 0$, as $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Finally, K is a linear combination of $\{h_i | i \in \hat{I}\}$.

• Step 2: Chevalley type relations

1) $[h_i, h_j] = 0$ are clear!

2) $[h_i, e_j] = a_{ij} \cdot e_j$ for all $i, j \in \hat{I}$
   - for $i, j \neq 0$, this follows from Chevalley rel-s for $\mathfrak{g}$
   - for $i = 0, j \neq 0$: $[h_0, e_j] = [K - h_\theta, e_j] = -[h_\theta, e_j] = \underbrace{-\alpha_j(h_\theta)}_{\text{our } a_{0j}} \cdot e_j$

   but let's express it solely through the pairing:

$$\boxed{a_{0j} = -\alpha_j(h_\theta) = (-\alpha_j, \theta^\vee) = (-\alpha_j, \theta)}$$

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(Continuation)

- for $i=0=j$: $[h_0, e_0] = [K - h_\theta, f_\theta t] = \theta(h_\theta)\cdot f_\theta t = (\theta, \theta^\vee) f_\theta t = 2\cdot f_\theta t$
- for $i\neq 0, j=0$: $[h_i, e_0] = [h_i, f_\theta t] = \underbrace{-\theta(h_i)}_{a_{i0}} \cdot f_\theta t$

and using the pairing we get: $a_{i0} = (-\theta, \alpha_i^\vee) = \frac{2}{(\alpha_i,\alpha_i)}\cdot(-(\alpha_i,\theta)) = -(\alpha_i,\theta)\cdot\frac{(\theta,\theta)}{(\alpha_i,\alpha_i)}$

[<u>Remark</u>. One has $\frac{(\theta,\theta)}{(\alpha_i,\alpha_i)} \in \{1,2,3\}$

3) The relations $[h_i, f_j] = -a_{ij} f_j$ $\forall i,j \in \hat{I}$ are verified analogously.

4) $[e_i, f_j] = \delta_{ij}\cdot h_i$ $\forall i,j\in\hat{I}$
- if $i,j\neq 0$, then it follows from such rel's in $\mathfrak{g}$
- if $i=0, j\neq 0 \Rightarrow [e_0, f_j] = [f_\theta t, f_i] \in \underbrace{\mathfrak{g}_{-\theta-\alpha_i}}_{\overset{\shortparallel}{0} \text{ as } \theta - \text{highest root}} \cdot t \Rightarrow [e_0, f_j] = 0$
- if $i\neq 0, j=0 \Rightarrow [e_i, f_0] = [e_i, e_\theta t^{-1}] \in \overset{\overset{0}{\shortparallel}}{\mathfrak{g}_{\theta+\alpha_i}}\cdot t^{-1} \Rightarrow [e_i, f_0] = 0$
- if $i=j=0$, then $[e_0, f_0] = [f_\theta t, e_\theta t^{-1}] = -h_\theta + K\cdot(f_\theta, e_\theta)$

it remains to show that $(f_\theta, e_\theta) = 1$. To this end, note:

$$(\underset{\overset{\shortparallel}{(-2f_\theta,\, e_\theta)}}{[h_\theta, f_\theta], e_\theta}) = (h_\theta, [f_\theta, e_\theta]) = (h_\theta, -h_\theta) = -(\theta^\vee, \theta^\vee) = -(\theta,\theta) = -2$$

$$\Rightarrow \boxed{(f_\theta, e_\theta) = 1}$$

This completes our verification of all Chevalley relations.

- <u>Moreover</u>: $\hat{A}$ is now fully recovered from A and $\boxed{\begin{array}{l} a_{0i} = -(\alpha_i,\theta) \\ a_{i0} = -(\alpha_i,\theta)\cdot\frac{(\theta,\theta)}{(\alpha_i,\alpha_i)} \end{array}}$

<u>Note</u>: $(\alpha_i,\theta)\geq 0$ as otherwise $\theta+\alpha_i$ would be a root (due to the property of root systems) contradicting θ-highest so that $\hat{A}$ is indeed generalized Cartan matrix

$\hat{A}$-indecomposable, which can be checked just case-by-case.

<u>Also</u>: If $\hat{D} := \mathrm{diag}(\frac{1}{d_0}, d_1, \ldots, d_n)$ with $d_i = \frac{(\alpha_i,\alpha_i)}{2}$, then $\hat{D}\cdot\hat{A}$ - symmetric

$(d_0\cdot a_{0j} = -(\alpha_j,\theta) = a_{j0}\cdot d_j,\ d_i\cdot a_{ij} = a_{ji} d_i = (\alpha_i,\alpha_j)$ for $i,j\neq 0)$

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(Continuation)

We claim that $\hat{D}\cdot\hat{A}$ defines a positive semi-definite form. To this end, we note that $\forall\ \vec{x}=\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}\in\mathbb{R}^{n+1}$ we have:

$$\boxed{\vec{x}^T\cdot\hat{D}\hat{A}\cdot\vec{x}=(x_0\cdot(-\theta)+x_1\alpha_1+\_\_+x_n\alpha_n,\ x_0\cdot(-\theta)+x_1\alpha_1+\_\_+x_n\alpha_n)\in\mathbb{R}_{\geq 0}}$$

Moreover, it's zero iff $\theta=\frac{x_1}{x_0}\alpha_1+\dots+\frac{x_n}{x_1}\alpha_n$. Evoking $\theta=a_1\alpha_1+\_\_+a_n\alpha_n$ with some $a_i\in\mathbb{Z}_{\geq 0}$ (actually $a_i\in\mathbb{Z}_{>0}$) we get that all $\vec{x}\in\mathbb{R}^{n+1}$ s.t. $\vec{x}^T\hat{D}\hat{A}\vec{x}=0$ are multiples of $\begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$. In particular, $\hat{D}\hat{A}$ is not positive definite.

To finish the proof, we need either to check Serre rel-s, which we leave as an exercise

Exercise: Verify Serre rel-s for $\{e_i\}$, $\{f_i\}$ $(i\in\hat{I})$

OR use the last condition of contragredient Lie algebras:

Step 3: Any graded ideal I of $\hat{\mathfrak{g}}$ intersects $\hat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}K$ nontrivially.

Assume contrary. Then it gives rise to graded nonzero ideal $\bar{I}$ in $L\mathfrak{g}=\hat{\mathfrak{g}}/K$. But if $\underset{\neq 0}{xt^n}\in\bar{I}$ then $[xt^n,\mathfrak{g}t^m]=\underset{\forall m}{\mathfrak{g}t^{n+m}}\Rightarrow\bar{I}=L\mathfrak{g}\Rightarrow\bar{I}\cap\mathfrak{h}t^0\neq 0\Rightarrow$ ↯.

Remark: The corresponding root lattice $\hat{Q}=\mathbb{Z}\alpha_0\oplus\mathbb{Z}\alpha_1\oplus\_\_\oplus\mathbb{Z}\alpha_n=\mathbb{Z}\alpha_0\oplus Q$ where Q = root lattice of $\mathfrak{g}$. Alternatively, if $\delta:=\alpha_0+\theta$, then $\hat{Q}=Q\oplus\mathbb{Z}\delta$ and δ generates the kernel of above pairing given by $\hat{D}\cdot\hat{A}$. In the loop realization, we get $\deg(xt^n)=\underbrace{\deg_\mathfrak{g}(x)}_{\in Q}+n\cdot\delta$ for any $x\in\mathfrak{g}$, $n\in\mathbb{Z}$. We also should have noted in the proof: $\hat{\mathfrak{g}}_0=\hat{\mathfrak{h}}$ has basis $\{h_i\}_{i=0}^n$, $\hat{\mathfrak{g}}_{\alpha_i}=\mathbb{C}e_i$, $\hat{\mathfrak{g}}_{-\alpha_i}=\mathbb{C}f_i$ $\forall i\in\hat{I}$

Remark: To fix the issue of $\delta\in\mathrm{Ker}$ (pairing), we shall be adding derivatives/degree generators next time.

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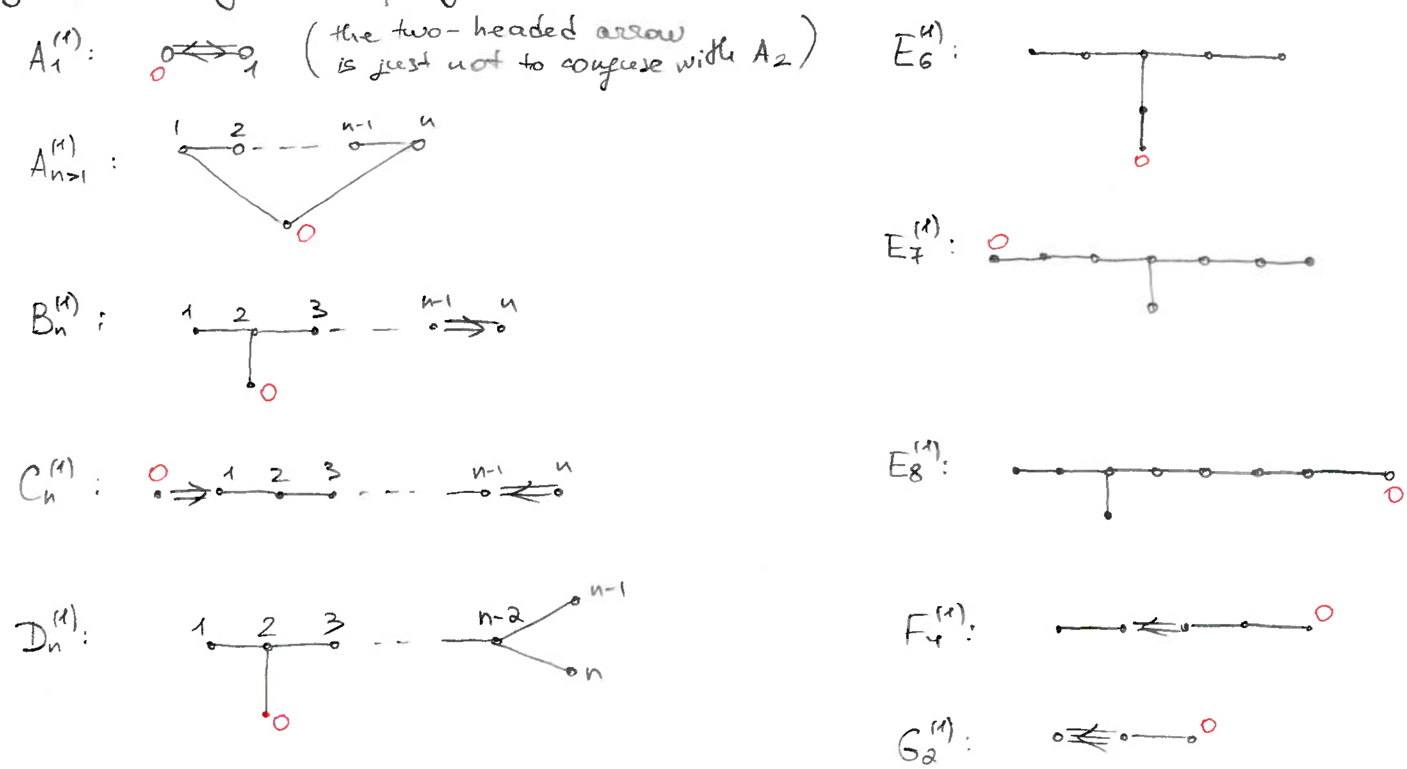
Let's now depict the corresponding extended/affine Dynkin diagrams. Evoking how $\hat{A}$ was expressed via A, it's clear we shall just add 1 vertex (labelled "0") and edges connecting it with others. If X_n is the Dynkin diagram of $\mathfrak{g}$, then we use $X_n^{(1)}$ for the above extended



Exercise: Prove the above pictures!

Remark: There are also twisted affine Kac-Moody algebras

$A_2^{(2)}$, $A_{2l}^{(2)}$ $(l \geq 2)$, $A_{2l-1}^{(2)}$ $(l \geq 3)$, $D_{l+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$

but we shall not discuss these in the class.

Next time: Category $\mathcal{O}$ and $\hat{\mathfrak{g}}$ instead of $\mathfrak{g}$

we recalled this for simple $\mathfrak{g}$ today, but I would rather add it to the next notes.