Lecture #22

Last time: If $\mathfrak{g}$-simple f.d. Lie algebra (with Cartan matrix A), then the algebra $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$ from Lecture 2 is affine Kac-Moody (with matrix $\hat{A}$ obtained from A by adding #0 row & column)

Remark: a) For simple $\mathfrak{g}$, taking $\mathfrak{g}(A^t)$ recovers another simple f.d. Lie algebra, called Langlands dual; which on the level of Dynkin diagrams reverses arrows

b) Reversing arrows on diagrams $B_n^{(1)}, C_n^{(1)}, F_4^{(1)}, G_2^{(1)}$ from Lecture 21 actually recovers Dynkin diagrams of twisted types $A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$)

c) According to [Kac, Thm 4.8] all affine Kac-Moody algebras are either as last time (denoted $X_n^{(1)}$), or those from b), or $A_2^{(2)}$ with diagram  or $A_{2n}^{(2)}$ with diagram 

Def 1 The roots of contragredient Lie algebra $\mathfrak{g}(A)$ are $\Delta := \{\alpha \in Q\backslash 0 \mid \mathfrak{g}(A)_\alpha \neq 0\}$.

Remark: a) The assignment $e_i \mapsto f_i,\ f_i \mapsto e_i,\ h_i \mapsto -h_i$ gives rise to a Lie alg. automorphism of $\tilde{\mathfrak{g}}(A)$ and hence of $\mathfrak{g}(A)$. In particular, have: $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{-\alpha})$.

b) For any $\alpha = \sum_{i=1}^{n} k_i\alpha_i$ $(k_i \in \mathbb{Z}_{\geq 0})$, the subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ is spanned

Exercise $\rightarrow$ by iterated commutators $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{N-1}}, e_{i_N}]\ldots]]$ with $\alpha_{i_1} + \ldots + \alpha_{i_N} = \alpha$. In particular, all $\mathfrak{g}_\alpha$ are indeed f.d. dim-l!

For the rest of today, we will crucially need the $\mathbb{C}$-vector space $F := Q \otimes_{\mathbb{Z}} \mathbb{C}$ (with basis $\alpha_1, \ldots, \alpha_n$) together with the natural linear map

$$\boxed{\begin{array}{ccc} F & \longrightarrow & \mathfrak{h}^* \\ \alpha & \longmapsto & \bar{\alpha} \end{array} \quad \text{s.t.} \quad \bar{\alpha}_j(h_i) = a_{ij} \quad \forall\, 1 \leq i,j \leq n}$$

This map can be also characterized as

$$\boxed{[h, x] = \bar{\alpha}(h)\cdot x \quad \forall x \in \mathfrak{g}_\alpha,\ h \in \mathfrak{h}}$$

In particular, this map is an isomorphism when A-nondegenerate, such as Cartan matrix of simple $\mathfrak{g}$. However, it's not for affine KM

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To clarify the last statement, recall that $\mathfrak{g}(\hat{A})$ is intrinsically graded by $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_n$ via $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$, $\deg(h_i) = 0$. Evoking that $\mathfrak{g}(\hat{A})$ can be realized as $\hat{\mathfrak{g}}$, it's natural to ask what this grading looks like in the loop realization. Recall $e_i \in \mathfrak{g}_{\alpha_i} \cdot t^0$, $f_i \in \mathfrak{g}_{-\alpha_i} t^0$, $h_i \in \mathfrak{h} \cdot t^0$, while $e_0 = f_\theta t \in \mathfrak{g}_{-\theta} \cdot t$, $f_0 = e_\theta t^{-1} \in \mathfrak{g}_\theta \cdot t^{-1}$, $h_0 = K - h_\theta$.

[Exercise: Show that $\forall x \in \mathfrak{g}_\beta$ $(\beta \in \Delta \cup \{0\})$, $\forall n \in \mathbb{Z}$: $\deg(x \cdot t^n) = \beta + n \cdot (\alpha_0 + \theta)$

Def 2: Let $\delta := \alpha_0 + \theta \in Q$

Then, we get:

$$\underset{\text{roots of } \hat{\mathfrak{g}}}{\hat{\Delta}} = \underset{\text{roots of } \mathfrak{g}}{\Delta} \times \mathbb{Z}\delta \amalg \{0\} \times (\mathbb{Z} \setminus 0)\delta \Rightarrow \hat{\Delta}_+ = \Delta_+ \amalg \left\{\alpha + k\delta \,\middle|\, \begin{array}{l} \alpha \in \Delta \cup \{0\} \\ k \in \mathbb{Z}_{>0} \end{array}\right\}$$

Note that $\bar{\delta} = 0 \in \mathfrak{h}^*$ as follows from Lecture #21. In particular:

Corollary 1: For untwisted affine Kac-Moody $\hat{\mathfrak{g}}$, $\mathrm{Ker}(F \xrightarrow[\alpha \mapsto \bar{\alpha}]{} \mathfrak{h}^*)$ is spanned by δ

- Our next goal is to introduce an appropriate category $\mathcal{O}$ of modules. We first start by reminding the classical definition for simple f.d.

Def 3: The category $\mathcal{O}$ of modules over simple f.d. $\mathfrak{g} = \mathfrak{g}(A)$ has

- Objects = $\mathfrak{g}$-modules M such that
  1) M is $\mathfrak{h}$-diagonalizable, i.e. $M = \bigoplus_{\mu \in \mathfrak{h}^*} M[\mu]$, $M[\mu] = \{v \in M \mid h(v) = \mu(h) \cdot v \ \forall h \in \mathfrak{h}\}$
  2) $\dim(M[\mu]) < \infty$
  3) $\mathrm{Supp}(M) \subseteq \bigcup_{i=1}^m D(\lambda_i)$ for some finite set $\{\lambda_1, \ldots, \lambda_m\} \subset \mathfrak{h}^*$

     where $\mathrm{Supp}(M) = \{\mu \in \mathfrak{h}^* \mid M[\mu] \neq 0\}$ — "support of M"

     $D(\lambda) := \{\lambda - k_1\bar{\alpha}_1 - \ldots - k_n\bar{\alpha}_n \mid k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}\}$
- Morphisms = $\mathfrak{g}$-module morphisms.

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Remark: The above definition of $\mathcal{O}$ is a refinement of the category $\mathcal{O}^+$ from Lecture 5, where instead of Q we used the principal grading with $\deg(e_i)=1=-\deg(f_i)$

As always, category $\mathcal{O}$ is designed to contain all Verma modules, as well as their $\oplus$, Ker, Coker.

Def 4: For $M\in\mathcal{O}$, define its character as

$$ch(M) := \sum_{\mu\in\mathfrak{h}^*} \dim(M[\mu])\cdot e^\mu \in \mathcal{R} := \left\{ \sum_{\nu\in\mathfrak{h}^*} a_\nu e^\nu \ \text{supported at finite union of } D(\lambda)\text{'s} \right\}$$

Exercise (standard!):

a) $ch(M_\lambda) = \dfrac{e^\lambda}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}$

b) $ch(M\otimes M') = ch(M)\cdot ch(M')$

c) for a short exact sequence $0\to N\to M\to M/N\to 0$ in category $\mathcal{O}$, have: $ch(M/N) = ch(M) - ch(N)$

• Category $\mathcal{O}$ for contragredient $\mathfrak{g}(A)$

! Warning: In general the Verma modules will not belong to $\mathcal{O}$, which poses the obvious issue, as $\mathcal{O}$ was exactly defined to contain all M_λ.

Indeed, if $x\in\mathfrak{g}(A)_\alpha$ satisfies $\bar{\alpha}=0$, then $h(xv_\lambda) = x\underbrace{hv_\lambda}_{\lambda(h)\cdot v_\lambda} + \underbrace{\bar{\alpha}(h)}_{=0}\cdot xv_\lambda$

$\Rightarrow \left\{ xv_\lambda \,\middle|\, \begin{array}{l}\deg(x)=0\\ \deg(x)\in-\Delta_+\end{array}\right\}$ are all in $M_\lambda[\lambda]$, and are all lin. indep.

E.g. for $\mathfrak{g}(\hat{A})=\hat{\mathfrak{g}}$, we can take these $x = h\cdot t^k$ $\forall h\in\mathfrak{h}\setminus 0$, $k\in\mathbb{Z}_{<0}$.

Thus, we obtain obvious issue to property 2) of category $\mathcal{O}$.

To get around, we shall now enlarge $\mathfrak{g}(A)$:

Def 5: Let $\mathfrak{g}_{ext}(A) := \mathfrak{g}(A)\oplus\mathbb{C}D_1\oplus\ldots\oplus\mathbb{C}D_n$ s.t. $[D_i,e_j]=\delta_{ij}$, $[D_i,f_j]=-\delta_{ij}$, $[D_i,h_j]=0$, $[D_i,D_j]=0$

In other words $\mathfrak{g}_{ext}(A) = (\mathbb{C}D_1\oplus\ldots\oplus\mathbb{C}D_n)\ltimes\mathfrak{g}(A)$ as D_i act via derivatives

$\Rightarrow$ triangular decomposition has same $\mathfrak{n}_\pm$, but $\mathfrak{h}_{ext} = \mathfrak{h}\oplus\mathbb{C}D_1\oplus\ldots\oplus\mathbb{C}D_n$.

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With this update $\mathfrak{h} \rightsquigarrow \mathfrak{h}_{ext}$ we shall now be able to introduce category $\mathcal{O}$.

First, we enlarge $F = Q \otimes_{\mathbb{Z}} \mathbb{C}$ to $\boxed{P := \mathfrak{h}^* \oplus F}$ and upgrade the previous map $F \to \mathfrak{h}^*$ to $P \xrightarrow{\varphi} \mathfrak{h}^*_{ext}$ via $\varphi(\alpha_j): h_i \mapsto a_{ij},\ D_i \mapsto \delta_{ij}$

$\varphi(\gamma): h \mapsto \gamma(h),\ D_i \mapsto 0 \quad \forall \gamma \in \mathfrak{h}^*$.

Lemma 1: φ is a vector space isomorphism

▸ The corresponding matrix (w.r.t. appropriate bases) is $\left(\begin{array}{c|c} I & * \\ \hline 0 & I \end{array}\right) \Rightarrow$ nondegenerate ▪

Note that we still have $[h, x] = \varphi(\alpha)(h) \cdot x \quad \forall h \in \mathfrak{h}_{ext} \quad \forall x \in \mathfrak{g}_\alpha$.

! With this upgrade, we can now define category $\mathcal{O}$ as before, whereas replacing $\mathfrak{h} \rightsquigarrow \mathfrak{h}_{ext}$ and $\mathfrak{h}^* \rightsquigarrow \mathfrak{h}^*_{ext}$. In particular, the Verma modules M_λ (and their irreducible quotients L_λ) are parametrized by $\lambda \in \mathfrak{h}^*_{ext}$.

Remark: In [Feigin-Zelevinsky], while they have the same P, their definition of category $\mathcal{O}$ is slightly different. Instead of viewing M as $\mathfrak{g}_{ext}(A)$-modules, they do work with $\mathfrak{g}(A)$-modules BUT add an extra data of P-grading on M

Exercise: Prove $ch(M_\lambda) = e^\lambda / \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\dim \mathfrak{g}_\alpha} \quad \forall \lambda \in P$

Note: Even for $A =$ Cartan matrix of simple f.d. $\mathfrak{g}$, the above definition looks quite different from the one we started with. However, this can be easily resolved by next lemma (see [Feigin-Zelevinsky, §2.8])

Lemma 2: For $\chi \in \mathfrak{h}^*$, let $\mathcal{O}_\chi$ be the full subcategory of $\mathcal{O}$ with weights in $\chi + F$

a) $\mathcal{O} = \bigoplus_{\chi \in \mathfrak{h}^*} \mathcal{O}_\chi$

b) If $\chi, \chi' \in \mathfrak{h}^*$ satisfy $\chi - \chi' = \bar{\alpha}$, then $\mathcal{O}_\chi$ & $\mathcal{O}_{\chi'}$ are naturally equivalent

c) If A is nonsingular, then every $\mathcal{O}_\chi$ is equiv. to $\mathcal{O}_0$ ($\mathcal{O}_0$ equals our $\mathcal{O}$ from Def 3 if $\mathfrak{g}(A)$-f.d.)

▸ a) is clear. c) follows from b). For b), given any $M' \in \mathcal{O}_{\chi'}$ we define $M \in \mathcal{O}_\chi$ which $= M'$ as $\mathfrak{g}(A)$-module, but $M_\nu = M'_{\nu - \chi + \chi' + \alpha}$ ▪

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Remark: For affine Kac-Moody $\mathfrak{g}(\hat{A}) = \hat{\mathfrak{g}}$, know $\operatorname{codim}(F \underset{\alpha \mapsto \bar{\alpha}}{\longrightarrow} \mathfrak{h}^*) = 1$. Hence, by Lemma 2(b), there is essentially 1-parameter family of categories $\mathcal{O}_k$ ($k \in \mathbb{C}$), where k = "level" = value on central elt $K \in \hat{\mathfrak{g}}$.

We shall end the class with the following result, needed for the next class:

Lemma 3: a) The center Z of contragredient $\mathfrak{g}(A)$ is
$$Z = \{\textstyle\sum_{i=1}^n \beta_i h_i \mid \beta_i \in \mathbb{C} \text{ s.t. } \sum \beta_i a_{ij} = 0 \ \forall j\}$$
(in particular, $\dim Z = \dim(\operatorname{Ker} A)$)

b) If $a_{ii} \neq 0 \ \forall i$ (in particular, if A - generalized Cartan matrix), then $[\mathfrak{g}', \mathfrak{g}'] = \mathfrak{g}'$ with $\mathfrak{g}' = \mathfrak{g}'(A)$

c) If A - indecomposable symmetrizable matrix, then any proper graded ideal of $\mathfrak{g} = \mathfrak{g}(A)$ is contained in its center Z. In particular, if A - nondegenerate, then $\mathfrak{g}(A)$ has no proper ideals.

▶ a) If $x \in Z \setminus \{0\}$, then $x = \sum_{\alpha \in Q} x_\alpha$, $\deg(x_\alpha) = \alpha$ and $x_\alpha \in Z$! (for degree reasons).
If $\exists \alpha \neq 0$ s.t. $x_\alpha \neq 0$, then $\mathbb{C}x_\alpha$ is a graded ideal of $\mathfrak{g}(A)$ non-intersecting $\mathfrak{h} \Rightarrow$ ↯
So $x \in \mathfrak{h} \Rightarrow$ can write $x = \sum \beta_i h_i$ (as $\{h_i\}$-basis) with $\beta_i \in \mathbb{C}$.
Then $x \in Z \Leftrightarrow [x, e_j] = 0 = [x, f_j] \ \forall j \Leftrightarrow \sum_i \beta_i a_{ij} = 0 \ \forall j$.

b) $[e_i, f_i] = h_i$, $[h_i, e_i] = a_{ii} e_i$, $[h_i, f_i] = -a_{ii} f_i$, $\{e_i, h_i, f_i\}$ - generate $\mathfrak{g}' \Rightarrow$ follows!

c) Assume $0 \subsetneq I \subsetneq \mathfrak{g}(A)$ is a graded ideal $\Rightarrow I = \bigoplus_{\alpha \in Q} I \cap \mathfrak{g}_\alpha \Rightarrow I = \underbrace{I_+}_{I \cap \mathfrak{n}_+} \oplus \underbrace{I_0}_{I \cap \mathfrak{h}} \oplus \underbrace{I_-}_{I \cap \mathfrak{n}_-}$

Claim: $I_+ = 0$ (and likewise $I_- = 0$).

▶ If not, pick $x \in I \cap \mathfrak{g}_\alpha$, $\alpha \in \Delta_+$, $x \neq 0$. Then the ideal J generated by x must intersect $\mathfrak{h}$ nontrivially. This implies that $\exists i_1, \ldots, i_r, j_1, \ldots, j_s$ s.t. Exercise!
$\operatorname{ad}(f_{i_1}) \ldots \operatorname{ad}(f_{i_r}) \operatorname{ad}(e_{j_1}) \ldots \operatorname{ad}(e_{j_s})(x) \in \mathfrak{h} \setminus \{0\}$
But $[f_{i_1}, \mathfrak{g}] \cap \mathfrak{h} = \mathbb{C} h_{i_1}$ $\Big\} \Rightarrow \underline{h_{i_1} \in J \subseteq I}$

Now $[h_{i_1}, e_j] = a_{i_1 j} e_j$, $[h_{i_1}, f_j] = -a_{i_1 j} f_j \Rightarrow \{e_j, f_j, h_j\} \in J \subseteq I$ if $a_{i_1 j} \neq 0$

Exercise: Use indecomposability of A to deduce $J = I = \mathfrak{g} \Rightarrow$ ↯ □

So: $I = I_0$ and if $\exists h \in I_0 \setminus Z \Rightarrow \exists j$ s.t. $[h, e_j] \neq 0 \Rightarrow e_j \in I \Rightarrow I_+ \neq 0 \Rightarrow$ ↯ ■