Lecture #24

Last time: A-symmetrizable indecomposable $\rightsquigarrow$ $\mathfrak{g}(A)$ - nondegenerate $\mathbb{Z}$-graded Lie alg.
$\rightsquigarrow$ $\mathfrak{g}^{ext}(A)$ has nondeg. symm. form.

Recall that for the case $\mathfrak{g}=\mathfrak{g}(A)$ - simple fin. dim. Lie algebra, the notion of the Casimir element plays often an important role. In this context, it is:

$$\boxed{C=\sum_{\substack{a\text{-orthonormal}\\ \text{basis of } \mathfrak{g}}} a^2 = \sum_{\alpha\in\Delta^+}(f_\alpha e_\alpha+e_\alpha f_\alpha)+\sum_{i=1}^{r=\dim\mathfrak{h}} x_i^2 = 2\sum_{\alpha\in\Delta^+} f_\alpha e_\alpha+\sum_{i=1}^{r} x_i^2+\underbrace{\sum_{\alpha\in\Delta^+} h_\alpha}_{=h_{2\rho}}}$$

where
- $e_\alpha\in\mathfrak{g}_\alpha$, $f_\alpha\in\mathfrak{g}_{-\alpha}$ are normalized to satisfy $(e_\alpha,f_\alpha)=1$, so that $[e_\alpha,f_\alpha]=h_\alpha$
- $\{x_i\}_{i=1}^r$ - orthonormal basis of $\mathfrak{h}$
- $2\rho:=\sum_{\alpha\in\Delta^+}\alpha$ is known to satisfy $\rho(h_i)=1$

Want: Generalize this to general Kac-Moody algebras

However, in this context Δ^+ may be an infinite set (and $\dim\mathfrak{g}_\alpha$ may be >1).
To overcome this, we shall rather introduce Casimir operator on any $M\in\mathcal{O}$

Def 1: Pick $\rho\in\mathfrak{h}^*$ so that $\rho(h_i)=\frac{a_{ii}}{2}=1$ $\forall i$.

Note: $(\rho,\alpha_k)=\rho(h_{\alpha_k})=d_k^{-1}\cdot\rho(h_k)=\frac{1}{d_k}=\frac{a_{kk}}{2d_k}=(\underbrace{d_k^{-1}h_k}_{=h_{\alpha_k}},\underbrace{d_k^{-1}h_k}_{=h_{\alpha_k}})\cdot\frac{1}{2}=\frac{(\alpha_k,\alpha_k)}{2}$

Recall: $(\ ,\ )$ - nondegenerate pairing (of degree 0) on $\mathfrak{g}^{ext}(A)$
$\Rightarrow$ restricts to non-deg. pairing $\mathfrak{g}_{-\alpha}\times\mathfrak{g}_\alpha\to\mathbb{C}$ $\forall\alpha\in\Delta^+\cup\{0\}$

- For any $\alpha\in\Delta^+$, let $\{e_\alpha^{(i)}\}_i$ - basis of $\mathfrak{g}_\alpha$, $\{f_\alpha^{(i)}\}_i$ - basis of $\mathfrak{g}_{-\alpha}$, s.t. $(e_\alpha^{(i)},f_\alpha^{(j)})=\delta_{ij}$ (note $\dim\mathfrak{g}_\alpha=\dim\mathfrak{g}_{-\alpha}<\infty$). We now define:

Def 2: For any $\mathfrak{g}(A)$-module M in category $\mathcal{O}$, define Casimir operator

$$\boxed{\begin{gathered}\Delta:=\Delta_+ +\Delta_0 \text{ acting on } M \text{ via:}\\ \Delta_+=2\sum_{\alpha\in\Delta^+}^{1\le i\le\dim\mathfrak{g}_\alpha} f_\alpha^{(i)}e_\alpha^{(i)},\qquad \Delta_0:=\sum_{x_i\text{- orthonormal basis of }\mathfrak{h}_{ext}} x_i^2+h_{2\rho}\end{gathered}}$$

While the sum in Δ_+ is infinite, its action on any $v\in M$ is well-defined!
Also: $\Delta_0|_{M[\mu]}=(\mu,\mu+2\rho)\cdot\mathrm{Id}$.

Theorem 1: a) The operator $\Delta: M \to M$ is a $\mathfrak{g}(A)$-module morphism.

b) On the Verma module: $\Delta_{|M_\lambda} = \mathrm{Id}\cdot(\lambda, \lambda+2\rho)$

▶ • a) $\Rightarrow$ b)

On $v_\lambda \in M_\lambda$ – highest weight vector: $\Delta(v_\lambda) = \Delta_0(v_\lambda) = (\lambda, \lambda+2\rho)\cdot v_\lambda$

As Δ commutes with $\mathfrak{g}(A)$-action, and M_λ is generated by v_λ, b) follows.

• Proof of a)

It suffices to verify Δ commutes with all the generators $\{e_k, f_k\}$.

Fix any $v \in M_{[\mu]}$. Then:

$$[\Delta_0, e_k](v) = \big((\mu+\alpha_k, \mu+\alpha_k+2\rho) - (\mu, \mu+2\rho)\big)\, e_k(v) = \big((\alpha_k,\alpha_k) + (2\mu+2\rho, \alpha_k)\big)\cdot e_k(v)$$

$$\underset{(\rho,\alpha_k)=\frac{1}{2}(\alpha_k,\alpha_k)}{=} (2\mu+2\alpha_k, \alpha_k)\cdot e_k(v) = 2\cdot h_{\alpha_k} e_k(v)$$

So: $\boxed{[\Delta_0, e_k] = 2h_{\alpha_k}\cdot e_k}$

On the other hand, we also have:

$$[\Delta_+, e_k] = 2\sum_{\alpha\in\Delta^+}^{i} [f_\alpha^{(i)} e_\alpha^{(i)}, e_k] = 2\sum_{\alpha\in\Delta^+}^{i} f_\alpha^{(i)} [e_\alpha^{(i)}, e_k] - 2\sum_{\alpha\in\Delta^+}^{i} [e_k, f_\alpha^{(i)}] e_\alpha^{(i)}$$

$$= -2h_{\alpha_k} e_k + 2\Big(\sum_{\alpha\in\Delta^+}^{i} f_\alpha^{(i)} [e_\alpha^{(i)}, e_k] - \sum_{\alpha\in\Delta^+\setminus\{\alpha_k\}}^{i} [e_k, f_\alpha^{(i)}] e_\alpha^{(i)}\Big)$$

Claim: If $\alpha, \alpha+\alpha_k \in \Delta^+$ and $1\le i\le \dim\mathfrak{g}_\alpha$, $1\le j\le \dim\mathfrak{g}_{\alpha+\alpha_k}$, then:

$$\sum_i f_\alpha^{(i)} \otimes [e_\alpha^{(i)}, e_k] = \sum_j [e_k, f_{\alpha+\alpha_k}^{(j)}] \otimes e_{\alpha+\alpha_k}^{(j)}$$

▷ $[e_\alpha^{(i)}, e_k] = \sum_j \big([e_\alpha^{(i)}, e_k), f_{\alpha+\alpha_k}^{(j)}\big)\, e_{\alpha+\alpha_k}^{(j)} \underset{\text{of pairing}}{\overset{\text{invariance}}{=}} \sum_j \big(e_\alpha^{(i)}, [e_k, f_{\alpha+\alpha_k}^{(j)}]\big)$

So: $\sum_i f_\alpha^{(i)} \otimes [e_\alpha^{(i)}, e_k] = \sum_i \sum_j f_\alpha^{(i)} \big(e_\alpha^{(i)}, [e_k, f_{\alpha+\alpha_k}^{(j)}]\big) \otimes e_{\alpha+\alpha_k}^{(j)}$

$$= \sum_j [e_k, f_{\alpha+\alpha_k}^{(j)}] \otimes e_{\alpha+\alpha_k}^{(j)}$$

□

Also note that $[e_\alpha^{(i)}, e_k] = 0$ if $\alpha+\alpha_k \notin \Delta^+$. Thus: $[\Delta, e_k] = 0$. Proof $[\Delta, f_k] = 0$ is similar.

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We shall now introduce a suitable version of replacement of finite dim. modules in the context of Kac-Moody Lie algebras $\mathfrak{g}(A)$.

Def 3: Given a Lie algebra $\mathfrak{g}$ and its module V, we say:

a) $v \in V$ is of *finite type* if $\dim(\mathcal{U}(\mathfrak{g})v) < \infty$

b) V is *locally finite* if every $v \in V$ is of finite type.

Exercise: Show that V is loc. finite iff it is a sum of fin. dim. $\mathfrak{g}$-modules (not direct sum!)

Def 4: A module V over a Kac-Moody algebra $\mathfrak{g}(A)$ is *integrable* if it is locally finite under each $\mathfrak{sl}_2^{(i)}$ (a copy of $\mathfrak{sl}_2$ spanned by e_i, f_i, h_i)

Note: Any loc. finite $\mathfrak{sl}_2$-module can be integrated to SL_2-module, hence, the name

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Proposition 1: $\mathfrak{g}(A)$ is an integrable $\mathfrak{g}(A)$-module w.r.t. adjoint action.

▷ Step 1: For any generator $x \in \{e_j, f_j\}$ and any i, $\mathcal{U}(\mathfrak{sl}_2^{(i)})(x)$ is fin. dim.

Let's take $x = f_j$ (the case $x = e_j$ is similar - do at home). For $i=j$, have $\mathfrak{sl}_2^{(j)}$-submodule generated by f_j is just $\mathfrak{sl}_2^{(j)}$, hence 3 dim! For $i \neq j$, we have $[e_i, f_j] = 0$, $[h_i, f_j]$ is a multiple of f_j, $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$ by Serre relations $\Rightarrow$ $\mathfrak{sl}_2^{(i)}$-submodule generated by f_j is again fin. dimensional!

Step 2: For any i, if $x, y \in \mathfrak{g}(A)$ are of finite type w.r.t. $\mathfrak{sl}_2^{(i)}$, then so is $[x,y]$. Indeed:

$$\mathcal{U}(\mathfrak{sl}_2^{(i)})([x,y]) \subseteq [\underbrace{\mathcal{U}(\mathfrak{sl}_2^{(i)})(x)}_{\text{fin. dim.}}, \underbrace{\mathcal{U}(\mathfrak{sl}_2^{(i)})(y)}_{\text{fin. dim}}] \leftarrow \text{fin. dim}$$

Since $\mathfrak{g}(A)$ is generated by e_j, f_j, we are done ◻

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Proposition 2: A $\mathfrak{g}(A)$-module V is integrable iff there is a set $\{v_s\}_{s\in S}$ of generators of V s.t. each v_s is of finite type w.r.t. each $\mathfrak{sl}_2^{(i)}$

► $\Rightarrow$: Obvious (just take all vectors of V)

$\Leftarrow$: It suffices to show that any $v \in U(\mathfrak{g}(A))v_s$ is of finite type w.r.t. $\mathfrak{sl}_2^{(i)}$ for any i. But $U(\mathfrak{g}) \simeq S(\mathfrak{g})$ as $\mathfrak{g}$-modules $\forall$ Lie algebra $\mathfrak{g}$ (by PBW thm)

Thus: $U(\mathfrak{g}(A))$ is loc. finite $\mathfrak{sl}_2^{(i)}$-module (indeed, $\mathfrak{g}(A)$ is loc. finite by Prop 1 $\Rightarrow$ so is $\mathfrak{g}(A)^{\otimes k}$ $\forall k$ $\Rightarrow$ so is $S(\mathfrak{g}(A)) \simeq U(\mathfrak{g}(A))$)

But then: $U(\mathfrak{g}(A))v_s$ is the image under action of $U(\mathfrak{g}(A)) \otimes W$ where $W = U(\mathfrak{sl}_2^{(i)})v_s$ - fin. dim, and tensor product preserves property "loc. finite". ▪

Proposition 3: L_λ - integrable $\Leftrightarrow$ $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$

► $\Rightarrow$: v_λ is of finite type $/\mathfrak{sl}_2^{(i)}$ $\xRightarrow[\mathfrak{sl}_2\text{-theory}]{}$ $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$.

$\Leftarrow$: By Proposition 2, it suffices to check that $v_\lambda \in L_\lambda$ is of fin. type $/\mathfrak{sl}_2^{(i)}$.

Consider $w_{\lambda,i} := f_i^{\lambda(h_i)+1} v_\lambda$. Then $e_j(w_{\lambda,i}) = 0$ $\forall j \neq i$ as $[e_j, f_i] = 0$.

Also, by $\mathfrak{sl}_2$-theory, if $j = i$ then $e_i(w_{\lambda,i}) = 0$

Thus, if $w_{\lambda,i} \neq 0$, then $U(\mathfrak{g}(A))(w_{\lambda,i})$ is a proper graded submodule of L_λ $\Rightarrow$ Contradiction. Thus, $w_{\lambda,i} = 0$ $\Rightarrow$ v_λ is of finite type over $\mathfrak{sl}_2^{(i)}$ ▪

Corollary 1: If $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$, then the $\mathfrak{g}(A)$-module $\tilde{L}_\lambda := M_\lambda / \langle f_i^{\lambda(h_i)+1} v_\lambda \rangle_i$ is integrable

► As above, it suffices to check v_λ is of fin. type $/\mathfrak{sl}_2^{(i)}$, which follows from $e_i(v_\lambda) = 0$, $f_i^{\lambda(h_i)+1} v_\lambda = 0$ in $\tilde{L}_\lambda$ ▪

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Def 5: The weight $\lambda \in P$ s.t. $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$ are called dominant integral. Let $P_+ \subseteq P$ be the set of all such weights.

Remark: For $\mathfrak{g} = \mathfrak{g}(A)$ - simple f.dim., we see that L_λ - integrable $\Leftrightarrow \dim(L_\lambda) < \infty$

Remark: We shall see next time that $\tilde{L}_\lambda \simeq L_\lambda$ $\forall \lambda \in P_+$.

Next time: Weyl-Kac character formula for $ch(L_\lambda)$ with $\lambda \in P_+$.