Lecture #25

Recall that $P = \mathfrak{h}^* \oplus F$ with $F = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}$ was equipped with symm. nondeg. pairing $(\, , \,) : P \times P \to \mathbb{C}$. We shall assume that $\mathfrak{g}(A)$ is Kac-Moody.

Def 1: a) For any i, define a linear map (called simple reflection) $r_i : P \to P$ via $\chi \mapsto \chi - \chi(h_i)\alpha_i$

b) The Weyl group of $\mathfrak{g}(A)$ is the subgroup W of $GL(P)$ generated by above r_i.

Exercise: a) $r_i^2 = \mathrm{Id}_P$

b) $(x, y) = (r_i x, r_i y) \quad \forall x, y \in P$

Remark: We have $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \ \forall i, j$. In particular, $F \subseteq P$ is W-invariant, while the quotient action $W \curvearrowright P/F$ is trivial. In fact, as we shall see next time, W can be viewed as a subgroup of $GL(F)$ for finite and affine Kac-Moody

Lemma 1: If V is an integrable $\mathfrak{g}(A)$-module, then $\forall \mu \in P$ there is a natural isomorphism $V[\mu] \xrightarrow{\sim} V[r_i(\mu)]$. In particular, $\dim V[\mu] = \dim V[r_i(\mu)]$

▸ As $V \in \mathcal{O}$, both $V[\mu], V[r_i(\mu)]$ are fin. dim. For any index i, due to integrability there is a fin. dim. $\mathfrak{sl}_2^{(i)}$-submodule W of V containing $V[\mu] + V[r_i(\mu)]$. Now by $\mathfrak{sl}_2$-theory, the result is immediate, in particular, $f_i^{\mu(h_i)} : V[\mu] \xrightarrow{\sim} V[r_i(\mu)]$ if $\mu(h_i) \in \mathbb{Z}_{\geq 0}$

$f_i^{-\mu(h_i)} : V[r_i(\mu)] \xrightarrow{\sim} V[\mu]$ if $\mu(h_i) \in \mathbb{Z}_{<0}$. ▮

Lemma 2: a) The form $(\, , \,)$ on P is W-invariant

b) If V-integrable, $\mu \in P$, $w \in W$, then $\dim V[\mu] = \dim V[w(\mu)]$

c) The set Δ of roots of $\mathfrak{g} = \mathfrak{g}(A)$ is W-invariant, and $\dim(\mathfrak{g}_\alpha) \overset{\forall w}{=} \dim(\mathfrak{g}_{w(\alpha)})$

d) $r_i(\alpha_i) = -\alpha_i$, and r_i permutes $\Delta_+ \setminus \{\alpha_i\}$

▶ a), b) – obvious from above

c) follows from $\mathfrak{g}(A)$ being an integrable module over itself (Prop 1 of Lect 24) so that $\mathrm{Supp}\, \mathfrak{g}(A) = \Delta \cup \{0\}$, and the result follows from b)

d) Clearly $r_i(\alpha_i) = \alpha_i - a_{ii}\alpha_i = -\alpha_i$

If $\beta \in \Delta^+ \setminus \{\alpha_i\}$, then $\beta = \sum k_j \alpha_j$ with $k_j \in \mathbb{Z}_{\geq 0}$ $\forall j$ and $\exists p \neq i$ s.t. $k_p \neq 0$.

Then $r_i(\beta) = \beta - * \cdot \alpha_i$ which has the same coeff $k_p > 0$ of $\alpha_p \Rightarrow r_i(\beta) \in \Delta^+$ ∎

Note that every $r_i : P \to P$ can be viewed as a reflection and has $\det = -1$.
This means that $\det : W \to \mathbb{C}$ actually take values in $\{\pm 1\}$ and

$$\boxed{\det(r_{i_1} \ldots r_{i_k}) = (-1)^k \quad \forall i_1, \ldots, i_k}$$

Now we are ready to state the key result of today:

Theorem 1 (Weyl-Kac character formula): Let $\lambda \in P_+$ and V be an integrable highest weight module of $\mathfrak{g}_{ext}(A)$. Then $V \simeq L_\lambda$ and

$$ch(V) = \sum_{w \in W} \det(w)\, ch\left(M_{w(\lambda+\rho)-\rho}\right) = \sum_{w \in W} \frac{\det(w) \cdot e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}$$

Before proving this result, let's first mention several corollaries:

Corollary 1: For $\lambda \in P_+$, $\tilde{L}_\lambda = L_\lambda$, cf. Corollary 1 of Lecture 24

This realizes L_λ as an explicit quotient of the Verma module

Corollary 2 (Weyl-Kac denominator formula): $\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \sum_{w \in W} \det(w) e^{w\rho - \rho}$

▶ Apply the theorem to $\lambda = 0$, whereas $L_0 = \mathbb{C}$ = trivial module ∎

This allows to rewrite the character above as:

$$\boxed{ch(L_\lambda) = \frac{\sum_{w \in W} \det(w)\, e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} \det(w)\, e^{w(\rho)-\rho}}}$$

Lecture #25

The proof of Theorem requires several "small" lemmas first.

As $W \curvearrowright P$, we shall define $w(e^{\mu}) := e^{w(\mu)}$.

$\boxed{\text{Lemma 3: } K := e^{\rho} \prod_{\alpha\in\Delta^+} (1-e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)} \text{ is } W\text{-anti-invariant, i.e. } w(K) = \det(w)\cdot K}$

► According to Lemma 2d): $r_i\left(\prod_{\alpha\in\Delta^+\setminus\{\alpha_i\}} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}\right) = \prod_{\alpha\in\Delta^+\setminus\{\alpha_i\}} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$

Also: $r_i(e^{\rho}(1-e^{-\alpha_i})) = e^{\rho-\rho(h_i)\alpha_i}(1-e^{\alpha_i}) = e^{\rho}\cdot e^{-\alpha_i}(1-e^{\alpha_i}) = -e^{\rho}(1-e^{-\alpha_i})$

$\Rightarrow r_i(K) = -K \;\; \forall i \Rightarrow w(K) = \det(w)\cdot K \;\; \forall w\in W$ ∎

Lemma 4: Let $\lambda\in P_+$ and recall $D(\lambda) := \{\lambda - \sum k_i\alpha_i \mid k_i\in\mathbb{Z}_{\geq 0}\}$

a) $W\lambda \subseteq D(\lambda)$

b) If $D \overset{\neq\emptyset}{\subseteq} D(\lambda)$ is a W-invariant set (i.e. $W(D)=D$), then $D\cap P_+ \neq \emptyset$

a) Know: L_λ - integrable (Prop 3 of Lecture 24) $\overset{\text{Lemma 2}}{\Longrightarrow}$ $\mathrm{Supp}(L_\lambda)$ is W-invariant.

Thus: $W\lambda \subseteq \mathrm{Supp}\, L_\lambda \subseteq \mathrm{Supp}\, M_\lambda = D(\lambda)$.

b) Pick any $\psi\in D$ and choose $w\in W$ so that $ht(\lambda - w\psi)$ is minimal, where for $\lambda - w\psi = \sum k_i\alpha_i$ it's height is $ht(\lambda - w\psi) = \sum_i k_i \in \mathbb{Z}_{\geq 0}$. We claim that $w\psi\in P_+$, as otherwise $w\psi(h_i)<0$ and $r_i(w\psi)$ would satisfy $ht(\lambda - r_i(w\psi)) < ht(\lambda - w\psi)$, a contradiction ∎

$\boxed{\text{Lemma 5: If } w\in W\setminus\{1\}, \text{ then } \exists i \text{ s.t. } w(\alpha_i)\in\Delta^-}$

► As $w\neq 1$, $\exists \lambda\in P_+$ s.t. $w(\lambda)\neq\lambda$. Then $w^{-1}(\lambda) = \lambda - \sum k_i\alpha_i$ with $k_i\in\mathbb{Z}_{\geq 0}$

by Lemma 4a) $\Rightarrow \lambda = w(w^{-1}(\lambda)) = \lambda - \sum l_i\alpha_i - \sum k_i w(\alpha_i)$ with $k_i, l_i\in\mathbb{Z}_{\geq 0}$

So: $\underbrace{\sum l_i\alpha_i}_{\neq 0} + \sum k_i w(\alpha_i) = 0 \Rightarrow \exists i$ s.t. $k_i>0$ and $w(\alpha_i)\in\Delta^-$ ∎

$\boxed{\text{Lemma 6: Let } \varphi, \psi\in P \text{ s.t. } \varphi(h_i)>0,\ \psi(h_i)\geq 0\ \forall i. \text{ Then } w\varphi=\psi \Rightarrow w=id}$

► If $w\neq id \Rightarrow \exists i$ s.t. $w(\alpha_i)\in\Delta^-$ by Lemma 5. Then:

$0 < (\varphi, \alpha_i) = (w^{-1}(\psi), \alpha_i) = (\psi, w(\alpha_i)) \leq 0 \Rightarrow$ ↯ ∎

Lemma 7: Let $\mu, \nu \in P_+$ be such that $\mu \in D(\nu) \setminus \{\nu\}$. Then:
$$(\nu+\rho, \nu+\rho) - (\mu+\rho, \mu+\rho) > 0.$$

▶ Write $\nu - \mu = \sum k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$ and some $k_i > 0$. Then:
$$(\nu+\rho,\nu+\rho)-(\mu+\rho,\mu+\rho) = (\nu-\mu, \nu+\mu+2\rho) = \sum_i \underbrace{k_i}_{\geq 0} \underbrace{(\alpha_i, (\nu+\rho)+(\mu+\rho))}_{>0} > 0$$
■

Finally, we shall need a refinement of the following result:

Lemma 8: For any $V \in \mathcal{O}$, one can write $ch(V) = \sum_{\alpha} c_\alpha \, ch(M_\alpha)$, where $c_\alpha \in \mathbb{Z}$, the sum is over $\alpha \in \bigcup_{\lambda \in Supp(V)} D(\lambda)$, and it can be infinite.

▶ As $V \in \mathcal{O}$, $\exists \lambda_1, \ldots, \lambda_m \in P$ such that $Supp(V) \subseteq \bigcup_{i=1}^{m} D(\lambda_i)$.

For any $\mu \in D(\lambda_i)$, we consider $ht_i(\mu) :=$ height of $\lambda_i - \mu$. Then, for any $\mu \in Supp(V)$, we define $h(\mu) := \sum_{i:\, \mu \in D(\lambda_i)} ht_i(\mu) \in \mathbb{Z}_{\geq 0}$. Let $h(V) := \min_{\mu \in Supp V} h(\mu)$.

Let $\mu_1, \ldots, \mu_s$ be all elements of $Supp(V)$ s.t. $h(\mu_1) = \ldots = h(\mu_s) = h(V)$, and let $\{v_{p,1}, \ldots, v_{p,t_p = \dim V[\mu_p]}\}$ be a basis of $V[\mu_p]$.

<u>Then</u>: each $v_{p,s} \in V$ is a highest weight vector by construction.

Hence, we get a nonzero $\mathfrak{g}(A)$-module morphism
$$\gamma : \bigoplus_{p=1}^{s} M_{\mu_p}^{\oplus t_p} \longrightarrow V.$$

Let $Ker = Ker(\gamma)$, $Coker = Coker(\gamma) = V/Im(\gamma)$, giving a long exact sequence
$$0 \to Ker \to \bigoplus M_{\mu_p}^{\oplus t_p} \to V \to Coker \to 0.$$

<u>Then</u>: $ch(V) = \sum t_p \cdot ch(M_{\mu_p}) + ch(Coker) - ch(Ker)$. But $Ker, Coker \in \mathcal{O}$, satisfy $Supp(Ker), Supp(Coker) \subseteq \bigcup_{i=1}^{m} D(\lambda_i)$, and finally $h(Ker), h(Coker) > h(V)$.

Applying the above to Ker or Coker instead of V, and proceeding alike, we get the result (note that it may not terminate!) ■

Lecture #25

Now we are ready to prove Theorem 1.

Proof of Thm 1

▶ Applying Lemma 8 to a highest weight λ integrable module V, we get $ch(V) = \sum_{\psi \in D(\lambda)} c_\psi \cdot ch(M_\psi)$ with $c_\psi \in \mathbb{Z}$, $c_\lambda = 1$. However, we have:

Lemma 9: If $c_\psi \neq 0$, then $(\psi+\rho, \psi+\rho) = (\lambda+\rho, \lambda+\rho)$

▷ As follows from the proof of Lemma 8, the Casimir operator Δ must act by the same constants on all M_ψ as on V, hence, $(\psi, \psi+2\rho) = (\lambda, \lambda+2\rho) \Rightarrow$ result □

Lemma 10: If $\psi+\rho = w(\lambda+\rho)$ for some $w \in W$, then $c_\psi = \det(w)$

▷ $ch(V) \cdot K = \sum_\psi c_\psi \cdot e^{\psi+\rho}$, but $\left.\begin{array}{l} ch(V) \text{ is } W\text{-invariant (Lemma 2b)} \\ K \text{ is } W\text{-anti-invariant} \end{array}\right\} \Rightarrow$

$\Rightarrow \sum c_\psi e^{\psi+\rho}$ is W-anti-invariant $\Rightarrow c_{w(\lambda+\rho)-\rho} = \det(w) \cdot c_\lambda = \det(w)$ □

Lemma 11: Let $D := \{\psi \in P \mid c_{\psi-\rho} \neq 0\}$. Then $D = W(\lambda+\rho)$

▷ By Lemma 10, know that $W(\lambda+\rho) \subseteq D$. Also by the proof, D is W-invariant subset of $\rho+D(\lambda)$. If $W(\lambda+\rho) \subsetneq D$, then by Lemma 4b) applied to $D \setminus W(\lambda+\rho)$, $\exists \beta \in (D \setminus W(\lambda+\rho)) \cap P_+$.

Now applying Lemma 7 (generalized to $\mu+\rho \in P_+$ instead of $\mu \in P_+$), get: $(\lambda+\rho, \lambda+\rho) > (\beta, \beta) \Rightarrow$ ↯ with Lemma 9. □

This completes the proof of the character formula for $ch(V)$, since the map $\begin{array}{l} W \longrightarrow P \\ w \longmapsto w(\lambda+\rho) \end{array}$ is injective by Lemma 6.

Finally, as $V \twoheadrightarrow L_\lambda$ and both V, L_λ - integrable $\Rightarrow ch(V) = ch(L_\lambda)$ $\Rightarrow V \xrightarrow{\sim} L_\lambda$ ■