HOMEWORK 1 (DUE JANUARY 28)

1. (a) Verify that the bilinear map $\omega\colon W \otimes W \to \mathbb{C}$ defined via $\omega(L_n, L_m) = (n^3 - n)\delta_{n,-m}$ defines a 2-cocycle on the Witt algebra W. Prove that it is not a 2-coboundary.

(b) Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a nontrivial invariant symmetric bilinear form $(\cdot,\cdot)$. Verify that the bilinear map $\omega\colon \mathfrak{g}[t,t^{-1}] \otimes \mathfrak{g}[t,t^{-1}] \to \mathbb{C}$ defined via $\omega(F,G) = \mathrm{Res}_{t=0}(dF, G)$ for $F, G \in \mathfrak{g}[t,t^{-1}]$ defines a 2-cocycle on the loop algebra $\mathfrak{g}[t,t^{-1}]$. Prove that it is not a 2-coboundary.

Note: This exercise completes our proofs of Theorems 1 and 2 from Lecture 2.

2. Consider the setup from Lecture 2: L is a Lie algebra, $\overline{L} \subset L$ is a Lie subalgebra, $M \subset L$ is an $\overline{L}$-submodule. Given a 2-cocycle $\omega \in Z^2(L)$, define $\varphi\colon \overline{L} \to M^*$ via

$$\varphi(x)(m) := \omega(x, m) \quad \forall m \in M.$$

Verify that $\varphi \in Z^1(\overline{L}, M^*)$, i.e. φ is a 1-cocycle.

Recall that a <u>representation</u> of a Lie algebra $\mathfrak{g}$ is a vector space V together with a linear map $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ s.t. $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)\ \forall x, y \in \mathfrak{g}$.

3. (a) Show that the Witt algebra W is a simple Lie algebra (i.e. it has no proper ideals).

(b) Deduce that W has no nontrivial finite dimensional representations.

4. For $\alpha, \beta \in \mathbb{C}$, let $V_{\alpha,\beta}$ be the vector space of formal expressions $g(t)t^\alpha(dt)^\beta$ with $g \in \mathbb{C}[t,t^{-1}]$ (*tensor fields* of rank β and branching α on the punctured complex plane $\mathbb{C}^\times$).

(a) Show that the formula

$$f\partial_t \circ g t^\alpha (dt)^\beta = (fg' + \alpha t^{-1} fg + \beta f' g) t^\alpha (dt)^\beta$$

defines an action of W on $V_{\alpha,\beta}$.

(b) Choose a basis $\{v_k\}_{k\in\mathbb{Z}}$ of $V_{\alpha,\beta}$ via $v_k := t^{k+\alpha}(dt)^\beta$. Verify

$$L_n(v_k) = -(k + \alpha + (n+1)\beta)v_{k+n}.$$