# HOMEWORK 2 (DUE FEBRUARY 4)

1. Recall the representations $V_{\alpha,\beta}$ of the Witt algebra $W$ from Homework 1.

(a) Find the necessary and sufficient conditions on $(\alpha, \beta, \alpha', \beta')$ under which $V_{\alpha,\beta} \simeq V_{\alpha',\beta'}$.

(b) Find the necessary and sufficient conditions on $(\alpha, \beta)$ under which $V_{\alpha,\beta}$ is irreducible.

2. Recall the Fock modules $F_\mu$ of the oscillator algebra $\mathcal{A}$ from Lecture 3.

(a) Construct an infinite-dimensional irreducible $\mathcal{A}$-representation, not isomorphic to any $F_\mu$.

(b) For any $\mathcal{A}$-representation $V$, let $V[0] = \{v \in V | K(v) = v, a_0(v) = \mu v, a_n(v) = 0 \, \forall n > 0\}$. Construct a natural $\mathcal{A}$-module homomorphism $F_\mu \otimes V[0] \to V$ and prove that it is injective.

3. (a) Consider the $\mathbb{Z}$-grading of $\mathfrak{g} = \mathfrak{sl}_n$ with $\deg(E_{ij}) = j - i$ (in particular, diagonal matrices are of degree 0). Verify that it is a non-degenerate $\mathbb{Z}$-graded Lie algebra.

(b) Consider the $\mathbb{Z}$-grading of $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ with $\deg(E_{ij}t^k) = j - i + nk, \deg K = 0$. Verify that it is a non-degenerate $\mathbb{Z}$-graded Lie algebra, while $\mathfrak{g}[t, t^{-1}]$ itself is not.

*Note:* Both statements hold for any simple finite dimensional $\mathfrak{g}$ and the principal grading.

4. (a) Let $\mathfrak{a}$ be a Lie algebra, $\mathfrak{b}$ be a Lie subalgebra of $\mathfrak{a}$, $M$ be a $\mathfrak{b}$-module, $N$ be an $\mathfrak{a}$-module. Prove that

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M) \otimes N \simeq \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M \otimes \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{a}}(N)) \quad \text{as} \quad \mathfrak{a}-\text{modules}.$$

(b) Let $\mathfrak{c}$ be a Lie algebra, $\mathfrak{a}, \mathfrak{b}$ be two Lie subalgebras of $\mathfrak{c}$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c}$. Note that $\mathfrak{a} \cap \mathfrak{b}$ is also a Lie subalgebra of $\mathfrak{c}$. Let $M$ be a $\mathfrak{b}$-module. Prove that

$$\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{c}}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{c}}(M)) \simeq \mathrm{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}}(\mathrm{Res}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{b}}(M)) \quad \text{as} \quad \mathfrak{a}-\text{modules}.$$

*Hint:* You may wish to use $U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b}) \simeq U(\mathfrak{c})$ both as left $\mathfrak{a}$ and right $\mathfrak{b}$ modules.

$5^\star$. Complete the proof of Proposition 3(b) from Lecture 3 by proving that the corresponding $\mathcal{A}$-module $\mathrm{Diff}(x_1, x_2, \ldots)/(\mathrm{Diff}(x_1, x_2, \ldots) \cdot I_v)$ is of finite length with all composition factors isomorphic to the Fock module $F_\mu$.

*Hint:* Construct a flag of subspaces $I_v = J_N \subset J_{N-1} \subset \ldots \subset J_1 \subset J_0 = \mathbb{C}[a_1, a_2, \ldots]$ such that $\dim(J_k/J_{k+1}) = 1$ and $a_\ell(J_k) \subset J_{k+1}$ for any $\ell \geq 1$ and $0 \leq k < N$, and consider the corresponding flag $D_N \subset D_{N-1} \subset \ldots \subset D_0 = \mathrm{Diff}(x_1, x_2, \ldots)$ with $D_k = \mathrm{Diff}(x_1, x_2, \ldots) \cdot J_k$.