HOMEWORK 3 (DUE FEBRUARY 11)

1. Is it true that any submodule of the Verma $\mathfrak{g}$-module M_λ is graded when $\mathfrak{g}$ is:

(a) the Heisenberg algebra (with $\deg a_k = k$);

(b) the Virasoro algebra (with $\deg L_k = k$);

(c) a finite-dimensional simple Lie algebra (with the principal $\mathbb{Z}$-grading).

2. Let $\phi\colon M_\lambda \to M_\mu$ be a nonzero homomorphism of Verma modules over a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ (with an abelian $\mathfrak{g}_0$). Prove that ϕ is injective.

Hint: You may wish to use the PBW theorem (stated in the form $\operatorname{gr} U(\mathfrak{n}_-) \simeq S(\mathfrak{n}_-)$).

3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ with a <u>real structure</u> $\dagger$. Define

$$\mathfrak{g}_{\mathbb{R}} := \{a \in \mathfrak{g} | a^\dagger = -a\}$$

It is usually called the <u>real form</u> of $\mathfrak{g}$, due to the following result:

(a) Prove that $\mathfrak{g}_{\mathbb{R}}$ is a Lie algebra over $\mathbb{R}$ and $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$ as Lie algebras over $\mathbb{C}$.

Assume now that $\mathfrak{g}$ is $\mathbb{Z}$-graded, $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$, and $\mathfrak{g}_k^\dagger = \mathfrak{g}_{-k}$.

(b) Verify that $\overline{(M^-_{-\lambda})^{-\dagger}} \simeq M^+_\lambda$ as modules over $\mathfrak{g}$ iff λ is <u>real</u> (that is, $\lambda \in \mathfrak{g}^*_{0,\mathbb{R}}$).

(c) Prove that M^+_λ has a nonzero hermitian form iff λ is real.

Note: Part (c) is exactly Lemma 4 from Lecture 6.

4. Recall the notion of the <u>character</u> for any module $V \in \mathcal{O}^+$ (Lecture 6):

$$\mathrm{ch}_V(q,x) := \sum_{d\in\mathbb{C}} q^{-d}\, \mathrm{Tr}_{V[d]}(e^x)$$

where q is a formal variable and $x \in \mathfrak{h} = \mathfrak{g}_0$. Prove the formula (stated in Lecture 6):

$$\mathrm{ch}_{M_\lambda}(q,x) = \frac{e^{\lambda(x)}}{\prod_{k>0} \det_{\mathfrak{g}_{-k}}(1 - q^k e^{\mathrm{ad}(x)})}$$

where the $\mathbb{C}$-grading on the Verma module M_λ is such that $\deg v^+_\lambda = 0$.

Hint: Use the PBW theorem ($M_\lambda \simeq S(\mathfrak{n}_-)$ as $\mathbb{Z}$-graded vector spaces) together with

$$\sum_{n\geq 0} q^n \mathrm{Tr}_{S^n V}(S^n A) = \frac{1}{\det(1 - qA)}$$

valid for any endomorphism A of a finite-dimensional vector space V.