HOMEWORK 4A (DUE FEBRUARY 25)

1. Let $\lambda, \mu \in \mathbb{C}$ and $\mathbf{i} := \sqrt{-1}$. Recall linear operators $\{\widetilde{L}_n\}_{n\in\mathbb{Z}}$ acting on F_μ from Lecture 7:

$$\widetilde{L}_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{n+j} + \mathbf{i}\lambda n a_n \quad \text{if } n \neq 0$$

$$\widetilde{L}_0 = \frac{\lambda^2+\mu^2}{2} + \sum_{j>0} a_{-j}a_j$$

(a) Verify the following equality in $\mathrm{End}(F_\mu)$ (for any $m, n \in \mathbb{Z}$):

$$[\widetilde{L}_n, a_m] = -m a_{n+m} + \mathbf{i}\lambda m^2 \delta_{m,-n}\mathrm{Id}$$

(b) Show that $\widetilde{L}_n$ define an action of Vir on F_μ with the central charge $c = 1 + 12\lambda^2$, i.e.

$$[\widetilde{L}_n, \widetilde{L}_m] = (n-m)\widetilde{L}_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}(1+12\lambda^2)$$

for any $m, n \in \mathbb{Z}$.

Note: This proves Proposition 1 of Lecture 7.

2. Let $\delta \in \{0, 1/2\}$. Recall the algebra C_δ (generated by the fermions $\{\psi_j\}_{j\in\delta+\mathbb{Z}}$) acting on the vector space V_δ (polynomials in anticommuting variables $\{\xi_j\}_{j\in\delta+\mathbb{Z}_{\geq 0}}$) from Lecture 8. We also recall linear operators $\{L_n\}_{n\in\mathbb{Z}}$ acting on V_δ via

$$L_n = \delta_{n,0}\frac{1-2\delta}{16} + \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j : \psi_{-j}\psi_{n+j} :$$

where the normal ordering is defined by

$$:\psi_i\psi_j: \; = \begin{cases} \psi_i\psi_j & \text{if } i \leq j, \\ -\psi_j\psi_i & \text{if } i > j. \end{cases}$$

(a) Verify the following equality in $\mathrm{End}(V_\delta)$ (for any $m \in \delta+\mathbb{Z}, n \in \mathbb{Z}$):

$$[\psi_m, L_n] = \left(m + \frac{n}{2}\right)\psi_{m+n}$$

(b) Show that L_n define an action of Vir on V_δ with the central charge $c = \frac{1}{2}$, i.e.

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{24}$$

for any $m, n \in \mathbb{Z}$.

Note: This proves Proposition 1 of Lecture 8.