HOMEWORK 4B (DUE FEBRUARY 25)

Given an associative algebra $\mathcal{C}$, a ($\mathcal{C}$-valued) quantum field is a formal bi-infinite power series

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \qquad \text{with } A_n \in \mathcal{C}$$

We split such $A(z)$ into two parts $A(z) = A_+(z) + A_-(z)$ via:[1]

$$A_+(z) = \sum_{n\leq -1} A_n z^{-n-1} \in \mathcal{C}[[z]] \;\text{ and }\; A_-(z) = \sum_{n\geq 0} A_n z^{-n-1} \in z^{-1}\mathcal{C}[[z^{-1}]]$$

Given two quantum fields $A(z), B(z) \in \mathcal{C}[[z, z^{-1}]]$, define their normally ordered product via:

$$: A(z)B(w) : = A_+(z)B(w) + B(w)A_-(z)$$

1. Let $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ be the quantum fields with coefficients in the universal enveloping of the Heisenberg ($\mathcal{A}$) and Virasoro (Vir) algebras, respectively.

(a) Compute the difference $a(z)a(w) - : a(z)a(w) :$ on the Fock representation F_μ. Present the corresponding power series by rational functions (depending only on $z - w$).

(b) Evaluate the difference $T(z)a(w) - : T(z)a(w) :$ on the Fock representation F_μ, viewed as a $\mathrm{Vir} \ltimes \mathcal{A}$-module. Present the answer as a linear combination of $a(w)$ and its derivatives with coefficients being rational functions in $z - w$.

(c) Evaluate the difference $T(z)T(w) - : T(z)T(w) :$ on the highest weight Vir representation with central charge c. Present the answer as a linear combination of $T(w)$ and its derivatives with coefficients being rational functions in $z - w$.

Recall the delta-function from Lecture 8:

$$\delta(w - z) = \sum_{n\in\mathbb{Z}} z^{-n-1} w^n = \frac{1}{z-w} + \frac{1}{w-z}$$

where $\frac{1}{z-w}$ denotes the power series expansion of the rational function in $|z| \gg |w|$:

$$\frac{1}{z-w} = \sum_{n\geq 0} z^{-n-1} w^n$$

2. Express $[a(z), a(w)]$, $[T(z), a(w)]$, $[T(z), T(w)]$ via $a(z), T(z), \delta(w - z)$ and its derivatives.

3*. Let F_0 be the Fock module of $\mathcal{A}$, $1 \in F_0$ denote the highest weight vector, $1 \in F_0^*$ denote the lowest weight vector of the dual representation, and $a(z)$ be as in Problem 1. Prove:

$$\left\langle 1^*, a(z_1)\cdots a(z_{2n})1 \right\rangle = \sum_{\{\sigma\in S_{2n}\,:\; \sigma^2=1,\, \sigma(i)\neq i \;\forall i\}} \prod_{i<\sigma(i)} \frac{1}{(z_i - z_{\sigma(i)})^2}$$

[1]Note that this is the only splitting compatible with $\partial_z \in \mathrm{End}\,\mathcal{C}[[z, z^{-1}]]$.

4*. This Problem outlines a proof of Theorem 1 from Lecture 4 stating that for a non-degenerate $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ and any $n \geq 0$, the restriction

$$(\cdot,\cdot)_\lambda \colon M^+_\lambda[-n] \times M^-_{-\lambda}[n] \longrightarrow \mathbb{C} \text{ is nondegenerate for generic } \lambda \in \mathfrak{h}^*$$

Identifying $M^\pm_{\pm\lambda}[\mp n] \simeq U(\mathfrak{n}_\mp)[\mp n]$ and choosing some fixed bases of the latter, this reduces to a non-vanishing of the corresponding determinant, denoted $\det(\cdot,\cdot)^{\mathfrak{g}}_{\lambda,n}$. The key idea will be to degenerate $\mathfrak{g}$ to a "generalized Heisenberg algebra" where the proof is more feasible.

Step 1 (degeneration process): Consider the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}^\epsilon = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}^\epsilon_n$ with $\mathfrak{g}^\epsilon_n = \mathfrak{g}_n$ as vector spaces, and with the Lie bracket defined via

$$[x,y]_\epsilon = [x,y] \cdot \epsilon^{1+\delta_{n,0}+\delta_{m,0}-\delta_{n+m,0}} \quad \text{for any } x \in \mathfrak{g}^\epsilon_n, y \in \mathfrak{g}^\epsilon_m$$

For $\epsilon \neq 0$, show that the following linear map is a Lie algebra isomorphism

$$\varphi_\epsilon \colon \mathfrak{g}^\epsilon \to \mathfrak{g} \qquad \text{with } x \mapsto \epsilon^{1+\delta_{n,0}} x \text{ for } x \in \mathfrak{g}^\epsilon_n$$

Show that

$$(xv^{+,\mathfrak{g}^\epsilon}_\lambda, yv^{-,\mathfrak{g}^\epsilon}_{-\lambda})_\lambda = (\varphi_\epsilon(x)v^{+,\mathfrak{g}}_{\lambda/\epsilon^2}, \varphi_\epsilon(y)v^{-,\mathfrak{g}}_{-\lambda/\epsilon^2})_{\lambda/\epsilon^2}$$

for any $x \in U(\mathfrak{n}_-), y \in U(\mathfrak{n}_+)$. Restricting to degree $\pm n$ components, deduce:

$$\det(\cdot,\cdot)^{\mathfrak{g}^\epsilon}_{\lambda,n} = \epsilon^N \det(\cdot,\cdot)^{\mathfrak{g}}_{\lambda/\epsilon^2,n}$$

for some $N \in \mathbb{Z}_{\geq 0}$.

Conclusion: Deduce that the leading term of $\det(\cdot,\cdot)^{\mathfrak{g}}_{\lambda,n} = \det(\cdot,\cdot)^{\mathfrak{g}^1}_{\lambda,n}$ equals $\det(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}$.

Therefore, it suffices to prove the non-vanishing of $\det(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}$ for generic $\lambda \in \mathfrak{h}^*$.

Step 2 (degenerated version explicitly): Note that $\mathfrak{g}^0 = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ as vector spaces with

$$[x,y]_{\mathfrak{g}^0} = \begin{cases} [x,y] & \text{if } \deg(x)+\deg(y)=0 \\ 0 & \text{otherwise} \end{cases}$$

for homogeneous elements x, y (hence, we call $\mathfrak{g}^0$ a "generalized Heisenberg algebra"). Note that $\mathfrak{n}_\pm = \bigoplus_{n>0} \mathfrak{g}^0_{\pm n}$ are abelian, so that $U(\mathfrak{n}_\pm) \simeq S(\mathfrak{n}_\pm)$.

Verify that the $\mathfrak{g}$-invariant form $(\cdot,\cdot)^{\mathfrak{g}^0}_\lambda \colon S(\mathfrak{n}_-) \times S(\mathfrak{n}_+) \to \mathbb{C}$ is given by

$$(a_1 \dots a_k, b_1 \dots b_l) = \delta_{k,l} \sum_{\sigma \in S(k)} \lambda([a_1, b_{\sigma(1)}]) \cdots \lambda([a_k, b_{\sigma(k)}]) \text{ with } \lambda_{|\mathfrak{g}_{\neq 0}} = 0 \tag{1}$$

Step 3 (verification for $\mathfrak{g}^0$): Use formula (1) to show that $\det(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n} \neq 0$ for generic $\lambda \in \mathfrak{h}^*$.