HOMEWORK 5 (DUE MARCH 4)

1. (a) Prove that $\Lambda^m V$ and $S^m V$ are irreducible representations of $\mathfrak{gl}_\infty$ for any $m \in \mathbb{N}$.

(b) More generally, show that the π-th Schur modules

$$S_\pi(V) = \mathrm{Hom}_{S_n}(\pi, V^{\otimes n}) \qquad \text{with} \qquad n \geq 1 \text{ and } \pi \in \mathrm{Irr}(S_n)$$

are all irreducible representations of $\mathfrak{gl}_\infty$.

Hint: You may wish to use the classical Schur-Weyl duality.

2. Verify that the action of $\mathfrak{gl}_\infty$ on $\Lambda^{\frac{\infty}{2},m} V$ constructed in the class is indeed an action.

Note: This proves Proposition 1 of Lecture 9.

3. Recall the linear map $\hat{\rho}\colon \bar{\mathfrak{a}}_\infty \to \mathrm{End}(\Lambda^{\frac{\infty}{2},m} V)$ from Lecture 10. Following our notations, we represent $A \in \bar{\mathfrak{a}}_\infty$ as a 2×2 block matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. For $A, B \in \bar{\mathfrak{a}}_\infty$, define $\alpha(A,B) \in \mathrm{End}(\Lambda^{\frac{\infty}{2},m} V)$ via

$$\alpha(A,B) := [\hat{\rho}(A), \hat{\rho}(B)] - \hat{\rho}([A,B])$$

Verify the following formula:

$$\alpha(A,B) = \mathrm{Tr}(A_{12}B_{21} - B_{12}A_{21}) \cdot \mathrm{Id}$$

Note: This proves Proposition 1 of Lecture 10.

4. For $\gamma, \beta \in \mathbb{C}$, recall the Lie algebra embedding $\overline{\varphi}_{\gamma,\beta}\colon W \hookrightarrow \bar{\mathfrak{a}}_\infty$ constructed in Lecture 10.

(a) Verify the following formula (with α as in Problem 3):

$$\alpha\left(\overline{\varphi}_{\gamma,\beta}(L_n), \overline{\varphi}_{\gamma,\beta}(L_m)\right) = \delta_{n,-m}\left(\frac{n^3 - n}{12} c_\beta + 2n h_{\gamma,\beta}\right)$$

where

$$c_\beta = -12\beta^2 + 12\beta - 2, \qquad h_{\gamma,\beta} = \frac{\gamma(\gamma + 2\beta - 1)}{2}$$

(b) According to part (a) (see also Lecture 10), we get a Lie algebra embedding

$$\varphi_{\gamma,\beta}\colon \mathrm{Vir} \hookrightarrow \mathfrak{a}_\infty$$

determined by

$$C \mapsto c_\beta K,\ L_n \mapsto \overline{\varphi}_{\gamma,\beta}(L_n) + \delta_{n,0} h_{\gamma,\beta} K$$

Hence, there is a natural action of Vir on $\Lambda^{\frac{\infty}{2},m} V$ (depending on $\gamma, \beta \in \mathbb{C}$). Verify that $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \cdots \in \Lambda^{\frac{\infty}{2},m} V$ is a Vir highest weight vector of the highest weight

$$\left(\frac{(\gamma - m)(\gamma + 2\beta - m - 1)}{2}, -12\beta^2 + 12\beta - 2\right).$$