HOMEWORK 6 (DUE MARCH 11)

1. Verify the second formula from Theorem 1 of Lecture 11:

$$\Gamma^*(u) = u^{-m} z^{-1} \exp\left(-\sum_{j>0} \frac{a_{-j}}{j} u^j \right) \exp\left(\sum_{j>0} \frac{a_j}{j} u^{-j} \right)$$

2. Let 1 (resp. ψ_0) be the highest weight vector of the bosonic space $\mathcal{B}^{(0)}$ (resp. fermionic space $\mathcal{F}^{(0)}$) and $\langle \cdot, \cdot \rangle$ be the contravariant form on that space.

(a) Compute the inner product $\langle 1, \Gamma(u_1) \cdots \Gamma(u_n) \Gamma^*(v_1) \cdots \Gamma^*(v_n) 1 \rangle$ by using the explicit "vertex operator" formula for $\Gamma(u), \Gamma^*(u)$.

(b) Compute analogous inner product $\langle \psi_0, X(u_1) \cdots X(u_n) X^*(v_1) \cdots X^*(v_n) \psi_0 \rangle$.

(c) Equating the results of parts (a) and (b), deduce the following identity:

$$\frac{\prod_{1 \le i < j \le n} (u_i - u_j) \cdot \prod_{1 \le i < j \le n} (v_i - v_j)}{\prod_{i,j=1}^{n} (u_i - v_j)} = (-1)^{\frac{n(n-1)}{2}} \det \left(\frac{1}{u_i - v_j} \right)_{i,j=1}^{n}$$

(d) Give an elementary proof of the identity from part (c).

3. Let d be the degree operator in the Fock space $\mathcal{B}^{(0)} = F_0$ (so that d multiplies each homogeneous element by its degree, where $\deg(x_i) = i$). Recall the operator $\Gamma(u, v)$ acting on $\mathcal{B}^{(0)}$ from Lecture 11:

$$\Gamma(u, v) = \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j} \right) \cdot \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j \right)$$

Prove the following equality of formal series:

$$\mathrm{Tr}_{\mathcal{B}^{(0)}} \left(\Gamma(u, v) q^d \right) = \prod_{n \ge 1} \frac{1 - q^n}{(1 - q^n u/v)(1 - q^n v/u)}$$

Hint: Compute first the trace of the operator $e^{\alpha x} e^{\beta \partial} q^{\gamma x \partial}$ on the space $\mathbb{C}[x]$.