## HOMEWORK 7 (DUE MARCH 18)

1. Recall the set $M(\infty) = \mathrm{Id} + \mathfrak{gl}_\infty$ and its subset $\mathrm{GL}(\infty) \subset M(\infty)$ consisting of invertible elements, introduced in Lecture 13.

(a) Show that the matrix multiplication makes $M(\infty)$ into a monoid and $\mathrm{GL}(\infty)$ into a group.

(b) Verify that the formula

$$A(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots) = (Av_{i_0}) \wedge (Av_{i_1}) \wedge (Av_{i_2}) \wedge \cdots$$

defines an action of the monoid $M(\infty)$ and a group $\mathrm{GL}(\infty)$ on $\mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2},m}V$.

2. For $\tau \in \mathcal{F}^{(0)} \backslash \{0\}$, show that $\tau \in \Omega$ iff $S(\tau \otimes \tau) = 0$ (proving Theorem 1 of Lecture 13).

*Hint:* Deduce this result from (or just apply similar arguments as in) its finite-dimensional counterpart established in Lecture 12.

3. Prove that $\tau = \tau(x_1, x_2, x_3, \cdots)$ satisfies the first nontrivial equation of the KP hierarchy

$$\Big( \left(\partial_{z_1}^4 + 3\partial_{z_2}^2 - 4\partial_{z_1}\partial_{z_3}\right) \tau(x-z)\tau(x+z) \Big)_{|z=0} = 0$$

if and only if the function

$$u := 2\partial_x^2 \log \tau(x, y, t, c_4, c_5, \cdots)$$

satisfies the KP equation

$$\frac{3}{4}\partial_y^2 u = \partial_x \left( \partial_t u - \frac{3}{2} u \cdot \partial_x u - \frac{1}{4}\partial_x^3 u \right)$$

4. Prove the formula of Lecture 14 for the coefficient of the leading power of $h$ in $\det_m(c, h)$:

$$K_m = \prod_{r,s \geq 1}^{rs \leq m} \Big( (2r)^s \cdot s! \Big)^{m(r,s)}$$

with

$$m(r,s) = p\,(m - rs) - p\,(m - r(s+1))$$

where $p(n)$ is the partition function.