HOMEWORK 8 (DUE MARCH 25)

1. Prove that there exists a unique extension of the action $\widehat{\mathfrak{gl}}_n \curvearrowright F^{(m)}$ to an action $\widetilde{\mathfrak{gl}}_n \curvearrowright F^{(m)}$ with $d(\psi_m) = 0$.

Note: This proves Lemma 3 of Lecture 15.

2. Establish an isomorphism $\widehat{\mathfrak{gl}}_n \simeq (\widehat{\mathfrak{sl}}_n \oplus \mathcal{A})/(K_1 - K_2)$, where $K_1 = (K, 0), K_2 = (0, K)$.

Note: This proves Lemma 5 of Lecture 15.

3. (a) Prove that $\mathcal{F} = \Lambda^{\frac{\infty}{2}} V$ is an irreducible representation of the Clifford algebra generated by $\{\hat{v}_j, \check{v}_j\}_{j\in\mathbb{Z}}$.

(b) Compute $\mathrm{Tr}_{\mathcal{F}}(q^{\mathbf{d}} z^{\mathbf{m}})$, where $\mathbf{m}$ is the operator multiplying elements of $\mathcal{F}^{(m)}$ by the number m, while $\mathbf{d}$ is the operator multiplying homogeneous elements by their degree, defined via:

$$\deg(\psi_0) = 0,\ \deg(\hat{v}_j) = j,\ \deg(\check{v}_j) = -j\,.$$

4. (a) Using the boson-fermion isomorphism $\mathcal{F} \simeq \mathcal{B}$, compute the answer to Problem 3(b) using the bosonic realization.

(b) Deduce the Jacobi triple product identity:

$$\prod_{n\geq 0} (1 - q^n z)(1 - q^{n+1} z^{-1})(1 - q^{n+1}) = \sum_{m\in\mathbb{Z}} (-z)^m q^{\frac{m(m-1)}{2}}$$

(c) Substitute $q = z^3$ to obtain the Euler's pentagonal identity:

$$\prod_{n\geq 1} (1 - z^n) = 1 + \sum_{k\geq 1} (-1)^k \left(z^{\frac{k(3k+1)}{2}} + z^{\frac{k(3k-1)}{2}} \right)$$