HOMEWORK 9 (DUE APRIL 1)

1. Let $\mathfrak{g}$ be a simple Lie algebra with an invariant non-degenerate pairing $(\cdot,\cdot)$. Prove that the Casimir element (with respect to $(\cdot,\cdot)$) acts on the Verma $\mathfrak{g}$-module M_λ as $(\lambda, \lambda+2\rho)\mathrm{Id}_{M_\lambda}$.

2. Does any highest weight $\widehat{\mathfrak{g}}$-module admit a $\widetilde{\mathfrak{g}}$-action extending that of $\widehat{\mathfrak{g}}$?

Note: For non-critical levels, this was discussed in Lecture 17.

3. Verify that if an admissible $\widehat{\mathfrak{g}}$-module M of non-critical level is unitary as a $\widehat{\mathfrak{g}}$-module, then it is also unitary as a $\mathrm{Vir} \ltimes \widehat{\mathfrak{g}}$-module (via the Sugawara construction).

Note: This proves Proposition 2 of Lecture 17.

4. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated to a simple Lie algebra $\mathfrak{g}$. For any $a \in \mathfrak{g}$, we set:

$$a[n] := at^n \in \widehat{\mathfrak{g}} \qquad \text{and} \qquad a(z) := \sum_{n\in\mathbb{Z}} a[n]z^{-n-1}$$

(a) For a highest weight $\widehat{\mathfrak{g}}$-representation V, show that $a(z)$ defines a linear map $V \to V((z))$.

(b) Let V have a highest weight vector v with $hv = 0$ for h in the Cartan subalgebra of $\mathfrak{g}$ and $Kv = kv$ ($k \in \mathbb{C}$). Evaluate $\langle v, a(z_1)b(z_2)v\rangle$ (as a rational function).

(c) In the setup of (b), evaluate $\langle v, a(z_1)b(z_2)c(z_3)v\rangle$.