HOMEWORK 11 (DUE APRIL 15)

1. Recall the notations $F = Q \otimes_{\mathbb{Z}} \mathbb{C}$, $P = \mathfrak{h}^* \oplus F$, the isomorphism $\varphi\colon P \xrightarrow{\sim} \mathfrak{h}^*_{\mathrm{ext}}$, and the symmetric invariant pairing $(\cdot,\cdot)\colon \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ from Lecture 21.

(a) Show that the kernel of the pairing $(\cdot,\cdot)$ coincides with the center Z of $\mathfrak{g}(A)$.

Define a linear map $\gamma\colon F \to \mathfrak{h}$ via $\alpha_i \mapsto h_{\alpha_i} := d_i^{-1} h_i$, and set:

$$h_\alpha := \gamma(\alpha) \quad \text{for all} \quad \alpha \in F.$$

(b) Verify $(h_\alpha, h) = \bar{\alpha}(h)$ for all $\alpha \in F, h \in \mathfrak{h}$.

(c) Verify $[x, y] = (x, y) \cdot h_\alpha$ for all $\alpha \in \Delta, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$.

(d) Deduce from (a, c) that $\mathfrak{g}(A)$ (with the principal $\mathbb{Z}$-grading) is non-degenerate $\mathbb{Z}$-graded.

Consider the inner product $\langle \cdot,\cdot \rangle\colon P \times P \to \mathbb{C}$ defined by:

$$\langle \varphi + \alpha, \psi + \beta \rangle = \varphi(h_\beta) + \psi(h_\alpha) + (h_\alpha, h_\beta).$$

(e) Identifying $P \simeq \mathfrak{h}^*_{\mathrm{ext}}$, show that the induced pairing $(\cdot,\cdot)\colon \mathfrak{h}_{\mathrm{ext}} \times \mathfrak{h}_{\mathrm{ext}} \to \mathbb{C}$ is given by

$$(h_{\alpha_i}, h_{\alpha_j}) = d_i^{-1} a_{ij},\ (D_i, h_{\alpha_j}) = (h_{\alpha_j}, D_i) = \delta_{ij},\ (D_i, D_j) = 0.$$

(f) Extend the invariant pairing on $\mathfrak{g}(A)$ to an invariant <u>non-degenerate</u> pairing on $\mathfrak{g}_{\mathrm{ext}}(A)$.

Note: Thus, we obtain a non-degenerate pairing on the extended version of $\mathfrak{g}(A)$.

2. Let M^+_λ (resp. M^-_λ) be the highest weight (resp. lowest weight) Verma module over a finite dimensional simple Lie algebra $\mathfrak{g}$. Let V be any $\mathfrak{h}$-diagonalizable module over $\mathfrak{g}$. Establish a vector space isomorphism:

$$\mathrm{Hom}_{\mathfrak{g}}(M^+_\lambda \otimes M^-_\mu, V) \simeq V[\lambda + \mu].$$

3. Let $\mathfrak{g}$ be a simple Lie algebra with the generators $\{e_i, f_i, h_i\}_{i=1}^r$. Set:

$$x[n] := x \cdot t^n \in L\mathfrak{g} \quad \text{for} \quad x \in \mathfrak{g}.$$

Find the defining relations between the elements $\Big\{e_i[n], f_i[n], h_i[n]\Big\}_{1 \le i \le r}^{n \in \mathbb{Z}}$ generating $L\mathfrak{g}$.

Note: This provides an explicit "loop" realization of $L\mathfrak{g}$ by generators and relations.