HOMEWORK 12 (DUE APRIL 22)

1. Recall the simple reflections $r_i \in \operatorname{End} P$ from Lecture 22, defined via:

$$r_i(\chi) = \chi - \chi(h_i)\alpha_i$$

Verify that $r_i^2 = \mathrm{Id}$ and also that

$$\langle r_i(x), r_i(y)\rangle = \langle x, y\rangle \qquad \forall x, y \in P$$

with respect to the pairing $\langle \cdot, \cdot \rangle \colon P \times P \to \mathbb{C}$ from Homework 11.

2. Let $\mathfrak{g}(A) = \widehat{\mathfrak{g}}$ be the affinization of a simple finite dimensional Lie algebra $\mathfrak{g}$.

(a) Define the category $\mathcal{O}$ over $\widetilde{\mathfrak{g}} := \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$. Explain why it is basically equivalent to the category $\mathcal{O}$ over $\mathfrak{g}_{\mathrm{ext}}(A)$ as defined in Lecture 21.

(b) For a module $V \in \mathcal{O}$ of level k, verify the following relation between the actions of the Casimir operator Δ (of Lecture 22) and the Sugawara operator L_0 (of Lecture 17) on V:

$$\Delta = 2(k + h^\vee)(L_0 + d)$$

3. Prove that every integrable module V of finite length in category $\mathcal{O}$ over a Kac-Moody algebra is a direct sum of irreducible modules L_λ ($\lambda \in P_+$).

Hint: Show that any short exact sequence $0 \to L_\lambda \to M \to L_\mu \to 0$ of integrable modules in the category $\mathcal{O}$ splits by utilizing the Casimir operator Δ from Lecture 22.

4. Show that Verma module M_λ over an extended Kac-Moody algebra $\mathfrak{g}_{\mathrm{ext}}(A)$ is irreducible for generic $\lambda \in P$. Specify a countable collection of hyperplanes outside of which it is true.

Hint: Use the Casimir operator.

5. Let V be a module over an extended Kac-Moody algebra $\mathfrak{g}_{\mathrm{ext}}(A)$ from category $\mathcal{O}$. Show that for a Weil generic $\lambda \in P$ (more precisely, for λ outside of a countable collection of hyperplanes) the module $M_\lambda \otimes V$ is semisimple, and describe its decomposition into irreducibles.