HOMEWORK 13 (DUE APRIL 29)

1. (a) Show that the action $LG \curvearrowright L\mathfrak{g}$ can be uniquely extended to the action $LG \curvearrowright L\mathfrak{g}\oplus\mathbb{C}d$ (with respect to the Lie algebra embedding $L\mathfrak{g} \hookrightarrow L\mathfrak{g}\oplus\mathbb{C}d$). Verify that:

$$LG \ni g(t)\colon d \mapsto d - tg'(t)g(t)^{-1} \in L\mathfrak{g}\oplus\mathbb{C}d\,.$$

(b) Show that there is a unique lift of the action in (a) to the action $LG \curvearrowright \widetilde{\mathfrak{g}} = L\mathfrak{g}\oplus\mathbb{C}d\oplus\mathbb{C}K$ (with respect to the Lie algebra epimorphism $\widetilde{\mathfrak{g}} \twoheadrightarrow L\mathfrak{g}\oplus\mathbb{C}d$ with $K\mapsto 0$) which preserves the non-degenerate pairing on $\widetilde{\mathfrak{g}}$. Verify that this action is explicitly given by:

$$\begin{aligned}
&g(t)\colon K \mapsto K,\\
&g(t)\colon a(t) \mapsto g(t)a(t)g(t)^{-1} + \mathrm{Res}_{t=0}\left(g'(t)a(t)g(t)^{-1}\right)dt\cdot K, \quad \text{for } a(t)\in L\mathfrak{g},\\
&g(t)\colon d \mapsto d - tg'(t)g(t)^{-1} - \frac{1}{2}\mathrm{Res}_{t=0}\left(tg'(t)g(t)^{-1}, tg'(t)g(t)^{-1}\right)\frac{dt}{t}\cdot K.
\end{aligned}$$

(c) Deduce the explicit formulas for the action of t_k, r_α in the Weyl group of affine $\mathfrak{sl}_2$ from Lecture 24:

$$\begin{aligned}
&r_\alpha\colon \alpha\mapsto -\alpha, \qquad K\mapsto K, \qquad d\mapsto d,\\
&t_k\colon \alpha\mapsto \alpha + 2k\cdot K, \qquad K\mapsto K, \qquad d\mapsto d - k\cdot\alpha - k^2\cdot K.
\end{aligned}$$

2. Derive the Weyl-Kac denominator formula in the case of affine $\mathfrak{sl}_2$ from the Jacobi triple product identity.

3. Let $\Theta_{n,m}(\tau,z) := \Theta_{n,m}(\tau,z,0)$ with τ in the upper half of the complex plane.

(a) Show that for a fixed τ, this is a holomorphic function in z.

(b) Relate $\Theta_{n,m}$ with $\Theta_{0,1}$.

(c) Find the zeros of $\Theta_{n,m}$ and their multiplicities.

Hint: Use the Jacobi triple product identity to factorize $\Theta_{0,1}$.

The following problem is quite technical, but we'll need it in Lecture 25.

4*. (a) Prove the following product formula for theta functions $\Theta_{n,m} = \Theta_{n,m}(\tau,z,u)$:

$$\begin{aligned}
\Theta_{n,m}\cdot\Theta_{n',m'} &= \sum_{j\in\mathbb{Z} \mod (m+m')\mathbb{Z}} d_j^{(m,m',n,n')}\Theta_{n+n'+2mj,m+m'},\\
d_j^{(m,m',n,n')} &:= \sum_{k\in\frac{m'n-mn'+2jmm'}{2mm'(m+m')}+\mathbb{Z}} q^{mm'(m+m')k^2}.
\end{aligned}$$

(b) Let $\lambda = md + \frac{n}{2}\alpha \in P_+,\ m \geq n \geq 0$. Use part (a) to prove:

$$\begin{aligned}
&\mathrm{ch}_{L_d}(h)\mathrm{ch}_{L_\lambda}(h) = \sum_{k \in I} \psi_{m,n,k}(q)\mathrm{ch}_{L_{d+\lambda-k\alpha}}(h),\\
&I := \left\{ k \in \mathbb{Z} | -\frac{m-n+1}{2} \leq k \leq \frac{n}{2} \right\},\\
&\psi_{m,n,k}(q) := \frac{f_k^{(m,n)}(q) - f_{n+1-k}^{(m,n)}(q)}{\varphi(q)},\\
&f_k^{(m,n)}(q) := \sum_{j \in \mathbb{Z}} q^{(m+2)(m+3)j^2 + ((n+1)+2k(m+2))j + k^2}.
\end{aligned}$$

(c) For m, n, k as in part (b), define $r := n+1, s := n+1-2k$ for $k \geq 0$ and $r := m-n+1, s := m-n+2+2k$ for $k < 0$. Prove:

$$\begin{aligned}
&\varphi(q) \cdot q^{-k^2} \cdot \psi_{m,n,k}(q) = A + B + C, \text{ where}\\
&A := 1 - q^{rs} - q^{(m+2-r)(m+3-s)},\\
&B := \sum_{j>0} q^{(m+2)(m+3)j^2 + ((m+3)r-(m+2)s)j} \left(1 - q^{2(m+2)sj+rs}\right),\\
&C := \sum_{j>0} q^{(m+2)(m+3)j^2 - ((m+3)r-(m+2)s)j} \left(1 - q^{2(m+2)(m+3-s)j + (m+2-r)(m+3-s)}\right).
\end{aligned}$$

(d) Use part (c) to provide an algebraic proof of the fact $\psi_{m,n,k}(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$.