HOMEWORK 14

1. Complete the proof of the main Theorem from Lecture 26 by proving the following results:

(a) Show that the leading term of $\det\left(\langle\cdot,\cdot\rangle^\eta\right)$ equals $\prod_{\alpha>0}\prod_{n\geq 1} h_\alpha^{P(\eta-n\alpha)}$, up to a nonzero constant factor.

(b) Let V be a finite dimensional space and $\{H_s\}_{s\in S}$ be a countable union of hyperplanes in V defined by linear functions $f_s \in \mathbb{C}[V]$. Let $F \in \mathbb{C}[V]$ be such that the zero set $Z(F) \subset V$ is contained in the union $\bigcup_{s\in S} H_s$. Show that F is a product of some linear functions f_s (possibly with multiplicities), up to a nonzero constant factor.

(c) For $\alpha \in \Delta$ with $(\alpha,\alpha) \neq 0$, establish the linear independence of the functions $\{\phi_\beta(\cdot)\}_{\beta\in\mathbb{Q}\alpha\cap Q^+}$ defined via $\phi_\beta(\eta) := P(\eta-\beta)$ for $\eta \in Q^+$.

2. **Vertex Operator Construction for $\widehat{\mathfrak{sl}}_2$**

Let a_i be the standard generators of the Heisenberg algebra $\mathcal{A}$. Let F_μ be the Fock representation over $\mathcal{A}$, and set $F := \bigoplus_{m\in\mathbb{Z}} F_{\sqrt{2}m}$. Define vertex operators on F:

$$X_\pm(u) := \exp\left(\mp\sqrt{2}\sum_{n<0}\frac{a_n}{n}u^{-n}\right)\exp\left(\mp\sqrt{2}\sum_{n>0}\frac{a_n}{n}u^{-n}\right)S^{\pm 1}u^{\pm\sqrt{2}a_0},$$

where S is the operator of shift $m \to m+1$ (cf. $\Gamma(u), \Gamma^*(u)$ of Lecture 11).

(a) Show that

$$X_a(u)X_b(v) = (u-v)^{2ab} : X_a(u)X_b(v) : \text{ for any } a,b \in \{\pm\}$$

(by abuse of notations, we identify $\pm$ with ± 1 above). In particular,

$$X_a(u)X_b(v) = X_b(v)X_a(u)$$

in the sense that the matrix elements of both sides are series in u, v which converge (but in different regions!) to the same rational functions (note that in the case of $\Gamma(u), \Gamma^*(u)$, there was a minus sign; thus, while $\Gamma(u), \Gamma^*(u)$ are "fermions", $X_+(u), X_-(u)$ are "bosons"!).

(b) Calculate $\left\langle 1, X_+(u_1)\cdots X_+(u_n)X_-(v_1)\cdots X_-(v_n)1\right\rangle$ for a highest weight vector $1 \in F_0$.

(c) Find the commutation relation between $X_\pm(u)$ and a_n.

(d) Show that the assignment

$$e(u) = \sum_{n\in\mathbb{Z}} e[n]u^{-n-1} \mapsto X_+(u),\ f(u) = \sum_{n\in\mathbb{Z}} f[n]u^{-n-1} \mapsto X_-(u),\ h[n] \mapsto \sqrt{2}a_n,\ K \mapsto \mathrm{Id}_F$$

defines an action of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ on F. Show that this is a level one highest weight representation of $\widehat{\mathfrak{sl}}_2$ with the highest weight 0 with respect to $\mathfrak{sl}_2$.

(e) Show that F is an irreducible $\widehat{\mathfrak{sl}}_2$-representation. Compute its character, i.e. $\mathrm{Tr}_F(e^{zh}q^{\mathsf{d}})$, where h is the generator of $\mathfrak{sl}_2$ and d is the degree operator defined by the conditions $\mathsf{d}(1) = 0$ and $[\mathsf{d}, x[n]] = nx[n]$ for any $x \in \mathfrak{sl}_2$, $n \in \mathbb{Z}$.

3. Use the explicit realization of the fundamental representation $L_d = L_{\omega_0}$ of Problem 2 to get a direct proof of the character formula (see Lecture 25):

$$\mathrm{ch}_{L_d}(h) = \frac{\Theta_{0,1}(\tau, z, u)}{\varphi(q)} \qquad \text{for} \qquad h = 2\pi i \left(\tfrac{z}{2}\alpha - \tau d + uK\right) .$$