

Lecture 1

59800

∞ -dim Lie alg-s.

01/19/2021

Basic Examples → The Heisenberg algebra
→ Virasoro algebra
→ Kac-Moody algebras.

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TTh 1³⁰-2⁴⁵

O.H. 3⁰⁰-4⁰⁰

↑

may include
technical computer.

Hwk: Each Thur → due next Thur.

No midterm/final

Self-contained.

Any big results about simple Lie algebras - will be explicitly stated.

Lie algebras

$$[a, b] \in V$$

V ,
↑
 \mathbb{C} -vector space

$$[\cdot, \cdot]: \Lambda^2 V \rightarrow V$$

↑ skew-symmetric

+ Jacobi identity

$$\boxed{0 = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]}$$

$\forall X, Y, Z$

Examples

1) A -associative alg. } $(A, [\cdot, \cdot])$ -Lie alg.
 $[a, b] = a \cdot b - b \cdot a$

2) A -algebra, Derivations of $A =$
 $\{d: A \rightarrow A \mid d(ab) = d(a)b + ad(b)\}$

Derivations form a Lie algebra, w.r.t. above commutator bracket (1)

3) \mathfrak{g} -Lie algebra
↑ vector space together w/ Lie bracket

\mathfrak{g} acts on \mathfrak{g} by derivations
adjoint
 $x \mapsto y \mapsto \text{ad}(x)(y) = [x, y]$

Jacobi: $\text{ad}(X)([Y, Z]) = [\text{ad}(X)Y, Z] + [Y, \text{ad}(X)Z]$



$\text{ad}(X)$ - derivation of the Lie algebra.

Def 1: The Heisenberg algebra (= the oscillator algebra) A , which as a vector space looks as follows:

with the Lie bracket given by $A = \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$

has a basis
 $a_n = (t^n, 0) \forall n \in \mathbb{Z}$
 $K = (0, 1)$

$[\underset{\substack{\uparrow \\ \mathbb{C}[t, t^{-1}]}}{f, \alpha}, \underset{\substack{\uparrow \\ \mathbb{C}[t, t^{-1}]}}{g, \beta}] = (0, \text{Res}_{t=0} g df)$

K -central $[K, a_n] = 0 \forall n$
 $[a_n, a_m] = (0, \text{Res}_{t=0} t^m d(t^n)) = (0, n \delta_{n,-m}) = n \delta_{n,-m} K$

Note: $\{a_n\}_{n \geq 0}$ - commute, $\{a_n\}_{n \leq 0}$ - commute

$[a_n, a_{-n}] = n \cdot K$

Def 2: The Witt algebra \mathbb{W} is the algebra of vector fields on \mathbb{C}^x , explicitly:

$$\{f(t)\partial_t \mid f(t) \in \mathbb{C}[t, t^{-1}]\}$$

with the bracket given by

$$[f\partial_t, g\partial_t] = (fg' - gf')\partial_t$$

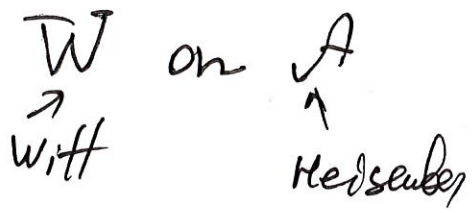
Down to earth, pick the basis $\{L_n = -t^{n+1}\partial_t \mid n \in \mathbb{Z}\}$

$$[L_n, L_m] = [t^{n+1}\partial_t, t^{m+1}\partial_t] = (m-n)t^{m+n+1}\partial_t = (n-m)L_{n+m}$$

$$[L_n, L_m] = (n-m)L_{n+m}$$

Rmk: Informally speaking, the \mathbb{R} -form of that ∞ -dim Lie algebra is a Lie alg. of the group of diffeomorphisms of S^1 $\mathbb{D}iff(S^1)$

Lemma 1: There is a natural action of W on A by derivations, i.e. we have



Lie alg. homomorphism

$$W \longrightarrow \text{Der } A$$

Rmk: Derivations are infinitesimal version of Lie alg. autom.

$$\{D: A \rightarrow A \mid D([a, b]) = [D(a), b] + [a, D(b)]\}$$

given by

$$\left(\frac{d}{dt} \right)_{\substack{\uparrow \\ W}} (g, d)_{\substack{\uparrow \\ A}} = \left(fg', 0 \right)_{\substack{\uparrow \\ A}}$$

$D: A \rightarrow A$ - derivation of Lie alg.

$$e^{tD} = 1 + t \cdot D + \frac{t^2 D^2}{2!} + \dots - \text{autom. of Lie alg. } A.$$

and other way around: given a 1-parameter family $\psi(t): A \rightarrow A$ of automorphisms of Lie alg A , the derivative $\psi'(0): A \rightarrow A$ is a derivation

Proof

1st part: each element of \mathbb{W} indeed acts by a derivation.

$$f \partial_t ([g, \alpha], [h, \beta]) \stackrel{?}{=} \underbrace{[(f \partial_t)(g, \alpha), [h, \beta]]}_{(0, \operatorname{Res}_{t=0} h dg)} + \underbrace{[g, \alpha], (f \partial_t)([h, \beta])}_{(0,0)}$$

$$(f \partial_t)(g, \alpha) = (f g', 0) \Rightarrow [(f g', 0), [h, \beta]] = (0, \operatorname{Res}_{t=0} h d(f g'))$$

$$2^{\text{nd}} \text{ summand} = [g, \alpha], (f h', 0) = (0, \operatorname{Res}_{t=0} f h' dg)$$

$$\text{Their sum} = (0, \operatorname{Res}_{t=0} \underbrace{h d(f g')}_{\cancel{f'g} + f'g'dt + fg''dt} + f h' dg) = (0, \operatorname{Res}_{t=0} (f'g'h + fg''h + f'g'h')) dt$$

$(f g' h)'$

$$= (0, \operatorname{Res}_{t=0} d(f g' h)) = (0, 0)$$

✓

2nd part of the proof : to see that it's Lie alg. homom.

Take two elements of W : $f \partial_t, g \partial_t$

Want :
$$\underbrace{[f \partial_t, g \partial_t]((h, \alpha))}_{\substack{\parallel \uparrow \\ W \\ (fg' - g'f) \partial_t}} \stackrel{?}{=} f \partial_t (g \partial_t (h, \alpha)) - g \partial_t (f \partial_t (h, \alpha))$$

def : Given two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ - Lie alg. hom.
 if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

LHS =
$$= (fg' - g'f) \partial_t (h, \alpha) = ((fg' - g'f)h, 0)$$

RHS =
$$f \partial_t (g h', 0) - g \partial_t (f h', 0) = (f (g h')', 0) - (g (f h')', 0)$$

$$= (f g' h' + \cancel{f g h''} - g f' h' - \cancel{g f h''}, 0)$$

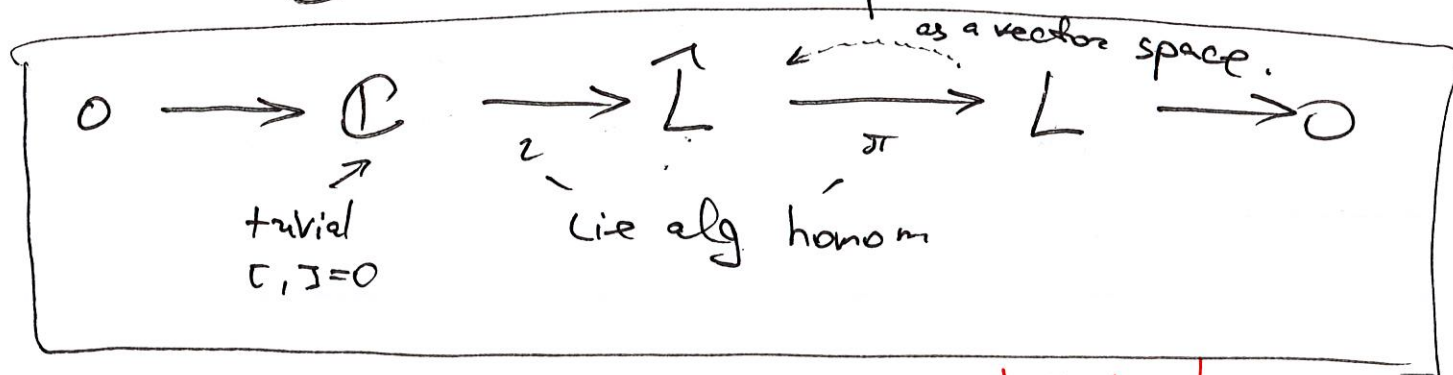


We will be interested a lot not in W itself but in its certain central extension, called Virasoro algebra.

Before we define it, let's talk about central extension (1-dim central extension).

Given Lie algebra L , a 1-dim central extension of L is a Lie alg. \hat{L} which fits in (short exact sequence):

! Must Request $\text{Im}(\iota)$ to be central in \hat{L} !



Split, as a vector space, $\hat{L} = L \oplus \mathbb{C}$ not unique!

any elt of \hat{L} can be encoded $(\overset{L}{a}, \overset{\mathbb{C}}{\alpha})$ no dependence on α, β !

the bracket on \hat{L} : $[(a, \alpha), (b, \beta)] = ([a, b], \underline{\omega(a, b)})$ (*)

Q: Which properties of ω are necess. & suff. to guarantee that we end up with Lie bracket on \hat{L} ?

A: (1) ω -skew-symmetric

(2) $\omega([a, b], c) + \omega([b, c], a) + \omega([c, a], b) = 0.$

Jacobi: $(a, \alpha), (b, \beta), (c, \gamma)$.

$$\tau(a, \alpha), \underbrace{\tau(b, \beta), (c, \gamma)}_{([b, c], \omega(b, c))} = \left(\underbrace{[a, [b, c]]}_{\text{bracket}}, \omega(a, [b, c]) \right)$$

+ 2 more = ~~~~~

~~Jacobi~~ $\Leftrightarrow (2)$.

UPSHOT: (*) defines a Lie bracket on \hat{L} iff ω satisfies (1, 2).

def: $Z^2(L) = \{\omega: (1, 2) \text{ hold}\} = \{2\text{-cocycles}\}$.

Q: Do different ω 's give equivalent central extensions?

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \hat{L}_1 & \xrightarrow{\omega} & L & \longrightarrow & 0 \\
 & & \downarrow \text{Id} & & \downarrow R & & \downarrow \text{Id} & & \\
 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \hat{L}_2 & \xrightarrow{\omega} & L & \longrightarrow & 0
 \end{array}
 \quad \begin{array}{l}
 \hat{L}_1 \sim \hat{L}_2 \\
 \text{1st equivalence relation}
 \end{array}$$

Down-to-earth, we just need to see how ω changes when we change the vector space splitting

$$\hat{L} = L \oplus \mathbb{C}$$

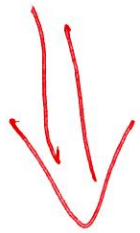
If those splittings differ by a linear map $L \xrightarrow{\xi} \mathbb{C}$
 $a \mapsto \xi(a)$

Easy computation: $\boxed{\omega_2(a, b) - \omega_1(a, b) = \xi([a, b])}$

$$\left(\begin{array}{l}
 ([a, \alpha], [b, \beta]) = ([a, b], \omega_1(a, b)) \quad \searrow \quad ([a, b], \omega_1(a, b) + \xi([a, b])) \\
 ([a, \alpha + \xi(a)], [b, \beta + \xi(b)]) = ([a, b], \omega_2(a, b))
 \end{array} \right)$$

Upshot: Changing the splitting $\hat{L} = L \oplus \mathbb{C}$
amount to identifying $\omega_1 \sim \omega_2$ s.t.

$$(\omega_1 - \omega_2)(a, b) = \underbrace{\frac{2}{\hbar}([a, b])}$$



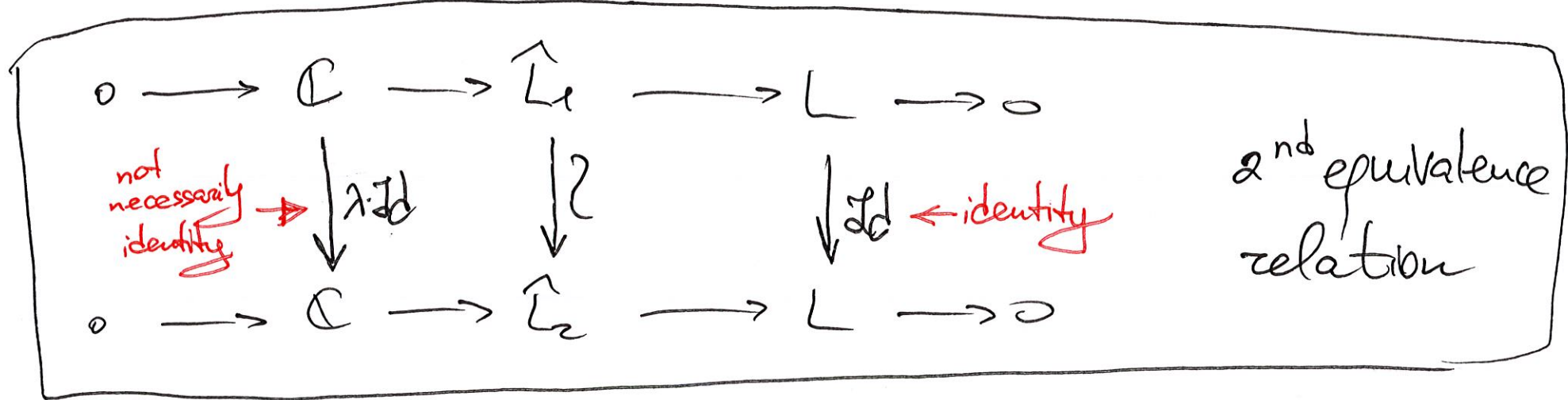
$B^2(L) =$ } such guys are called {
2-coboundaries }

Main Upshot:

The 1-dim central extensions
are parametrized by the

quotient $\frac{Z^2(L)}{B^2(L)} =: H^2(L) \leftarrow$ 2nd cohomology
of L .

If we slightly modify this notion of $\hat{L}_1 \sim \hat{L}_2$:



Then: non-trivial \mathbb{C} -dim central extensions of the Lie algebra L are parametrized by

$\mathbb{P}(H^2(L))$

projectivization of $H^2(L)$

Thm 1 (1) The space $H^2(W)$ is 1-dim!
with alg



there is a unique (up to our second) equivalence nontrivial central extension of \bar{W} .

2) The generator ω can be chosen as follows:

$$\omega(L_n, L_m) = (n^3 - n) \delta_{n, -m}.$$

Def 3: The Virasoro algebra, denoted Vir , is the central extension of \bar{W} defined by the 2-cocycle

$$\omega(L_n, L_m) = \frac{n^3 - n}{12} \delta_{n, -m}$$

(at the moment this rescaling isn't important).

Down-to-earth,

V_{ir} has basis $\{L_n\}_{n \in \mathbb{Z}} \cup \{C\}$ with

the bracket:

$$[C, L_n] = 0$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12} \delta_{n,-m} \cdot C$$

Disclaimer

In the discussion of central extensions, when looking at

$$0 \rightarrow \mathbb{C} \xrightarrow{z} \hat{L} \xrightarrow{\pi} L \rightarrow 0$$

we MUST request $\text{Im}(z)$ to be central in \hat{L} !

E.g. take $\mathfrak{a} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq \mathfrak{gl}_2$ - Lie algebra of upper-triangular matrices

pick $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{a}$

$$\begin{aligned} \text{Then: } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ - not central}$$

$$\text{BUT } [\mathfrak{a}, \mathbb{C} \cdot x] \subseteq \mathbb{C} \cdot x$$

\Downarrow

$$\text{have SES } 0 \rightarrow \mathbb{C} \cdot x \xrightarrow{z} \mathfrak{a} \xrightarrow{\pi} \mathfrak{a}/\mathbb{C} \cdot x \rightarrow 0$$

and $\text{Im}(z)$ is not central!