Lecture 1

59800 ∞-dim Lie alg-s.

01/19/2021

Basic Examples → The Heisenberg algebra
→ Virasoro algebra
→ Kac-Moody algebras.

Sasha

TTh $1^{30}-2^{45}$

O.H $3^{00}-4^{00}$

may include technical computat.

Hwk: Each Thur → due next Thur.

No midterm/final

Self-contained.

Any big results about simple Lie algebras. - will be explicitly stated.

Lie algebras

$[a,b] \in V$

V, $[\cdot,\cdot] : \Lambda^2 V \longrightarrow V$ + Jacobi identity

$\mathbb{C}$-vector space

skew-symmetric

$$\boxed{0 = [x,[y,z]] + [y,[z,x]] + [z,[x,y]]}$$

$\boxed{\forall x, y, z}$

Examples :

1) A - associative alg.
   $[a,b] = a \circ b - b \circ a$
   $\Big\}$ $(A, [\,,\,])$ - Lie alg.

2) A - algebra, Derivations ~~of~~ A =
   $\{ d : A \to A \mid d(ab) = d(a) \cdot b + a\, d(b) \}$

   Derivations form a Lie algebra, w.r.t. above commutator bracket in 1)

3) $\mathfrak{g}$ - Lie algebra
   vector space together w/ Lie bracket

   $\mathfrak{g} \xrightarrow[\text{adjoint}]{\text{acts}} \mathfrak{g}$ by derivations

   $x \qquad y \mapsto \mathrm{ad}(x)(y) = [x,y]$

Jacobi: $\mathrm{ad}(X)([Y,Z]) = [\mathrm{ad}(X)Y, Z] + [Y, \mathrm{ad}(X)Z]$

$$\Downarrow$$

$\boxed{\mathrm{ad}(X) \text{ – derivation of the Lie algebra.}}$

Def 1: The Heisenberg algebra (= the oscillator algebra) $\mathcal{A}$, which as a vector space looks as follows:

$$\boxed{\mathcal{A} = \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}}$$

has a basis
$a_n = (t^n, 0)$ $\forall n \in \mathbb{Z}$
$K = (0,1)$

with the Lie bracket given by

$$\boxed{[\underbrace{(f,\alpha)}_{f \in \mathbb{C}[t,t^{-1}]}, (g,\beta)] = (0, \operatorname{Res}_{t=0} g\,df)} \qquad \alpha, \beta \in \mathbb{C}$$

K – central $\quad [K, a_n] = 0 \quad \forall n.$

$[a_n, a_m] = (0, \operatorname{Res}_{t=0} t^m d(t^n)) = (0, n\delta_{n,-m}) = n \cdot \delta_{n,-m} \cdot K$

Note: $\{a_n\}_{n\geq 0}$ – commute, $\{a_n\}_{n\leq 0}$ – commute $\quad [a_n, a_{-n}] = n \cdot K.$

Def 2: The Witt algebra W is the algebra of vector fields on $\mathbb{C}^\times$, explicitly:

$$\boxed{\{f(t)\partial_t \mid f(t) \in \mathbb{C}[t, t^{-1}]\}},$$

with the bracket given by

$$\boxed{[f\partial_t, g\partial_t] = (fg' - gf')\partial_t}$$

Down to earth, pick the basis $\{L_n = -t^{n+1}\partial_t \mid n \in \mathbb{Z}\}$

$$[L_n, L_m] = [-t^{n+1}\partial_t, -t^{m+1}\partial_t] = (m-n)t^{m+n+1}\partial_t$$

$$= (n-m)\cdot L_{n+m}$$

$$\boxed{[L_n, L_m] = (n-m)L_{n+m}}$$

Rmk: Informally speaking, the $\mathbb{R}$-form of that ∞-dim Lie algebra is a Lie alg. of the group $\boxed{\mathrm{Diff}(S^1)}$ of diffeomorphisms of S^1

Lemma 1: There is a natural action of $\overline{W}$ on $\mathcal{A}$
Witt ↗ ↑ Heisenberg
by derivations, i.e. we have Lie alg. homomorphism

$$\boxed{\overline{W} \longrightarrow \underbrace{\mathrm{Der}\,\mathcal{A}}}$$

$\{D: \mathcal{A} \to \mathcal{A} \mid D([a,b]) = [D(a), b] + [a, D(b)]\}$

Rmk: Derivations are infinitesimal version of Lie alg. autom.

given by

$$\boxed{(\underbrace{f\partial_t}_{\in W})\,(\underbrace{g, \alpha}_{\in \mathcal{A}}) = (\underbrace{fg', 0}_{\in \mathcal{A}}).}$$

any Lie alg.

$D: \mathcal{A} \to \mathcal{A}$ - derivation of Lie alg.

$e^{tD} = 1 + t\cdot D + \frac{t^2 D^2}{2!} + \ldots$ - autom. of Lie alg. $\mathcal{A}$.

and other way around: given a 1-parameter family $\varphi(t): \mathcal{A} \to \mathcal{A}$ of automorphisms of Lie alg $\mathcal{A}$, the derivative $\varphi'(0): \mathcal{A} \to \mathcal{A}$ is a derivation

Proof

1st part: each element of $\overline{W}$ indeed acts by a derivation.

$$\underbrace{f\partial_t\big(\underbrace{[(g,\alpha),(h,\beta)]}_{(0,\ \mathrm{Res}_{t=0} h\,dg)}\big)}_{(0,0)} \overset{?}{=} \boxed{[(f\partial_t)(g,\alpha),(h,\beta)]} + [(g,\alpha),(f\partial_t)(h,\beta)].$$

$$(f\partial_t)(g,\alpha) = (fg',0) \Rightarrow [(fg',0),(h,\beta)] = (0, \mathrm{Res}_{t=0}\, h\,d(fg'))$$

$$2^{\text{nd}} \text{ summand} = [(g,\alpha),(fh',0)] = (0, \mathrm{Res}_{t=0}\, fh'\,dg)$$

$$\text{Their sum} = (0, \mathrm{Res}_{t=0}\, h\,\underbrace{d(fg')}_{\cancel{f'g}\ f'g'dt + fg''dt} + fh'\underbrace{dg}_{g'dt}) = (0, \mathrm{Res}_{t=0} \underbrace{(f'g'h + fg''h + fg'h')}_{(fg'h)'}dt)$$

$$= (0, \mathrm{Res}_{t=0}\, d(fg'h)) = (0,0).$$

✓

$\boxed{2^{nd} \text{ part of the proof}}$: to see that it's Lie alg. homom.

Take two elements of W : $f\partial_t, g\partial_t$

Want : $\underbrace{[f\partial_t, g\partial_t]}_{\in W}((h,\alpha)) \stackrel{?}{=} f\partial_t(g\partial_t(h,\alpha)) - g\partial_t(f\partial_t(h,\alpha))$

$\| \quad (fg'-g'f)\partial_t$

Def: Given two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ – Lie alg. hom. if $\varphi([x,y]) = [\varphi(x), \varphi(y)]$.

LHS $=$

$= (fg'-g'f)\partial_t(h,\alpha) = ((fg'-g'f)h, 0)$

RHS $= f\partial_t(gh', 0) - g\partial_t(fh', 0) = (f(gh')', 0) - (g(fh')', 0)$

$= (fg'h' + \cancel{fgh''} - gf'h' - \cancel{gfh''}, 0)$

$\checkmark$

7

We will be interested a lot not in W itself ~~but~~ in its certain central extension, called Virasoro algebra.

Before we define it, let's talk about central extension (1-dim central extension).

Given Lie algebra L, a 1-dim central extension of L is a Lie alg. $\hat{L}$ which ~~fits~~ in SES: (short exact sequence)

$$\boxed{\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r, "\iota"'] & \hat{L} \arrow[r, "\pi"'] & L \arrow[r] \arrow[l, dashed, bend right, "\text{as a vector space}"'] & 0 \end{tikzcd}}$$

$\mathbb{C}$: trivial $[\,,\,]=0$

ι, π: Lie alg homom

! Must Request $\mathrm{Im}(\iota)$ to be central in $\hat{L}$!

Split, as a vector space, $\hat{L} = L \oplus \mathbb{C}$ — not unique!

any elt of $\hat{L}$ can be encoded $(\underset{\in L}{a}, \underset{\in \mathbb{C}}{\alpha})$

the bracket on $\hat{L}$: $\boxed{[(a,\alpha),(b,\beta)] = ([a,b], \omega(a,b))}$ (*)

no dependence on α, β!

<u>Q</u>: Which properties of ω are necess. & suff. to guarantee that we end up with Lie bracket on $\hat{L}$?

<u>A</u>: (1) ω - skew-symmetric

(2) $\omega([a,b],c) + \omega([b,c],a) + \omega([c,a],b) = 0$.

Jacobi: $(a,\alpha), (b,\beta), (c,\gamma)$.

$$[(a,\alpha), \underbrace{[(b,\beta),(c,\gamma)]}_{([b,c],\,\omega(b,c))}] = ([a,[b,c]], \omega(a,[b,c]))$$

$+$ 2 more $= \ldots$

Jacobi $\Leftrightarrow$ (2).

UPSHOT: $(*)$ defines a Lie bracket on $\hat{L}$ iff ω satisfies (1,2).

$$\boxed{\text{Def: } Z^2(L) = \{\omega : (1,2) \text{ hold}\} = \{2\text{-cocycles}\}.}$$

Q: Do different ω's give equivalent central extensions?

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{Id}"] & \hat{L}_1 \arrow[r] \arrow[d, "\wr"] & L \arrow[r] \arrow[d, "\mathrm{Id}"] \arrow[l, bend right, "\ell"'] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & \hat{L}_2 \arrow[r] & L \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

$\hat{L}_1 \sim \hat{L}_2$

1st equivalence relation

Down-to-earth, we just need to see how ω changes when we change the vector space splitting

$$\boxed{\hat{L} = L \oplus \mathbb{C}.}$$

If those splittings differ by a linear map $L \xrightarrow{\xi} \mathbb{C}$, $a \mapsto \xi(a)$

Easy Computation: $\boxed{\omega_2(a,b) - \omega_1(a,b) = \xi([a,b])}$

$$\left(\begin{array}{l} [(a,\alpha),(b,\beta)] = ([a,b], \omega_1(a,b)) \rightsquigarrow ([a,b], \omega_1(a,b) + \xi([a,b])) \\ [(a,\alpha+\xi(a)),(b,\beta+\xi(b))] = ([a,b], \omega_2(a,b)) \end{array} \right)$$

Upshot: Changing the splitting $\hat{L} = L \oplus \mathbb{C}$ amount to identifying $\omega_1 \sim \omega_2$ s.t.

$$(\omega_1 - \omega_2)(a,b) = \underbrace{\xi([a,b])}$$

$\Downarrow$

$$\boxed{B^2(L) = \left\{ \begin{array}{c} \text{such guys are called} \\ \text{2-coboundaries} \end{array} \right\}}$$

Main Upshot: The 1-dim central extensions are parametrized by the quotient $\frac{Z^2(L)}{B^2(L)} =: H^2(L) \leftarrow$ 2nd cohomology of L.

If we slightly modify this notion of $\hat{L}_1 \sim \hat{L}_2$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\lambda \cdot Id"] & \hat{L}_1 \arrow[r] \arrow[d, "\wr"] & L \arrow[r] \arrow[d, "Id"] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & \hat{L}_2 \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

not necessarily identity → $\lambda \cdot Id$

Id ← identity

2nd equivalence relation

Then: non-trivial 1-dim central extensions of the Lie algebra L are parametrized by

$$\underbrace{\mathbb{P}(H^2(L))}_{\text{projectivization of } H^2(L)}$$

Thm 1: 1) The space $H^2(\underset{\text{with alg}}{\underbrace{W}})$ is 1-dim!

$\Downarrow$

there is a unique (up to our second equivalence) nontrivial central extension of W.

2) The generator ω can be chosen as follows:

$$\boxed{\omega(L_n, L_m) = (n^3 - n)\,\delta_{n,-m}.}$$

Def 3: The Virasoro algebra, denoted Vir, is the central extension of W defined by the 2-cocycle

$$\boxed{\omega(L_n, L_m) = \frac{n^3-n}{12}\,\delta_{n,m}}$$ (at the moment this rescaling isn't important).

Down-to-earth.

Vir has basis $\{L_n\}_{n\in\mathbb{Z}} \cup \{C\}$ with the bracket:

$$\boxed{\begin{aligned} &[C, L_n] = 0 \\ &[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\cdot C \end{aligned}}$$

Disclaimer

In the discussion of central extensions, when looking at

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \hat{L} \xrightarrow{\pi} L \longrightarrow 0$$

we <u>MUST</u> request $\mathrm{Im}(\iota)$ to be central in $\hat{L}$!

E.g. take $\mathfrak{a} = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \subseteq \mathfrak{gl}_2$ - Lie algebra of upper-Δ matrices

pick $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{a}$

<u>Then</u>: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ $\Rightarrow$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ - not central

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$

BUT $[\mathfrak{a}, \mathbb{C}\cdot x] \subseteq \mathbb{C}\cdot x$

$\Downarrow$

have SES $0 \to \mathbb{C}x \xrightarrow{\iota} \mathfrak{a} \xrightarrow{\pi} \mathfrak{a}/\mathbb{C}x \to 0$

and $\mathrm{Im}(\iota)$ is <u>not</u> central!