Lecture #2

A - Heisenberg alg.

$[a_n, a_m] = n\,\delta_{n,-m}\cdot K$

can rescale   central

W - Witt alg.

$[L_n, L_m] = (n-m)L_{n+m}$

Vir - Virasoro,

$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}\cdot K$

up to preference choice   central

2-cocycles $Z^2(L) = \{\omega : L\otimes L \to \mathbb{C} \mid \omega\text{ - skew},\ \omega([a,b],c)+\omega([b,c],a)+\omega([c,a],b)=0\}$

$\cup\!|$

2-coboundaries $B^2(L) = \{\omega : L\otimes L \to \mathbb{C} \mid \exists \xi \in L^* : \omega(a,b) = \xi([a,b])\}$

2-cohomology $\boxed{H^2(L) = Z^2(L)/B^2(L)}$

! Must: Impose $\mathrm{Im}(\iota)$ - central in $\hat{L}$.

$H^2(L) \longleftrightarrow$ 1-dim central extension of Lie alg. L up to equiv:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r, "\iota"] \arrow[d, "\mathrm{Id}"] & \hat{L}_1 \arrow[r, "\pi"] \arrow[d, "\wr"] & L \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \hat{L}_2 \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

$\mathbb{P}(H^2(L)) \longleftrightarrow$ 1-dim non-trivial central extension of Lie alg L up to equiv

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "\lambda\cdot\mathrm{Id}", "\lambda\neq 0"'] & \hat{L}_1 \arrow[r] \arrow[d, "\wr"] & L \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & L_2 \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

Thm 1 : $H^2(W) =$ 1-dim (one can choose a generator given by $\omega(L_n, L_m) = \frac{n^3-n}{6} \delta_{n,-m}$)

$$H^2(W) = Z^2(W) / B^2(W).$$

- Pick $\beta \in Z^2(W)$

Step 1 : Consider $\xi \in W^*$ s.t. $\xi(L_n) = \frac{1}{n}\beta(L_n, L_0)$ $\forall n \neq 0$

$\rightsquigarrow$ replace β by $\tilde{\beta}$ given by $\tilde{\beta}(a,b) = \beta(a,b) - \underbrace{\xi([a,b])}_{\text{2-coboundary}}$.

Key property : $\boxed{\tilde{\beta}(L_n, L_0) = 0 \quad \forall n \in \mathbb{Z}}$ $\quad [L_n, L_0] = n L_n$.

Step 2 : Let's write down 2-cocycle for $\tilde{\beta}$ & $a = L_0$, $b = L_m$, $c = L_n$ $\quad$ will vary.

$$\underbrace{\underbrace{\tilde{\beta}([L_0, L_m], L_n)}_{= m \cdot \tilde{\beta}(L_n, L_m)} + \underbrace{\tilde{\beta}([L_n, L_0], L_m)}_{= n \tilde{\beta}(L_n, L_m)} + \underbrace{\tilde{\beta}([L_m, L_n], L_0)}_{= 0 \text{ by Step 1}} = 0}_{(m+n)\tilde{\beta}(L_n, L_m) = 0 \quad \forall n, m \in \mathbb{Z}.}$$

So : $\boxed{\tilde{\beta}(L_n, L_m) = \delta_{n,-m} \cdot b_n, \quad b_n \in \mathbb{C}}$ $\longleftarrow$ $\boxed{b_n \overset{\tilde{\beta}\text{-skew}}{=} -b_{-n}}$

Step 3: Let's write 2-cocycle cond-n for $a = L_m$, $b = L_n$, $c = L_p$.

If $m+n+p \neq 0$, then it's vacuous (by previous step).

~~If~~ suffices to treat $m+n+p=0$, i.e. $p = -m-n$.

$$\underbrace{\tilde{\beta}(\underbrace{[L_m, L_n]}_{(m-n)L_{m+n}}, \underset{-m-n}{L_p})}_{(n-m)b_p} + \underbrace{\tilde{\beta}(\underbrace{[L_n, L_p]}_{(n-p)L_{n+p}}, L_m)}_{(p-n)b_m} + \underbrace{\tilde{\beta}([L_p, L_m], L_n)}_{(m-p)b_n} = 0$$

$n+p=-m$, $\quad p-n = -2n-m$, $\quad m-p = n+2m$

$$\boxed{(n-m)b_{n+m} = (2m+n)\,b_n - (m+2n)\,b_m}$$

Last Step: Change by 2-coboundary to achieve $b_1 = 0$.

Consider: $\overline{\beta}(a,b) = \tilde{\beta}(a,b) - \frac{b_1}{2}\xi([a,b])$, $\quad \xi \in W^*$ which extracts ~~coeff~~ coeff-t of L_0.

$$\overline{\beta}(L_1, L_{-1}) = b_1 - \frac{b_1}{2}\xi(\underbrace{[L_1, L_{-1}]}_{2L_0}) = 0$$

$\Rightarrow$ can assume $\boxed{b_1 = 0.}$

$b_1 = 0$

$(n-m)\, b_{n+m} = (2m+n)\, b_n - (m+2n)\, b_m.$

Pley $m \mapsto 1$, $n \mapsto n-1$

$(n-2)\, b_n = (n+1)\, b_{n-1}$

$$b_n = \frac{n+1}{n-2} \cdot \frac{n}{n-3} \cdots \frac{4}{1}\, b_2$$

$$= \boxed{\frac{n^3-n}{6}\, b_2}$$

Affine Kac-Moody algebras

- $\mathfrak{g}$ – fin. dim. Lie algebra
- $(\,,\,) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ – invariant form

$$\boxed{([x,y],z) + (y,[x,z]) = 0.}$$

$\Leftarrow$ (side remark: infinitesimal version of group-inv. form $(gx, gy) = (x,y)$)

$$\rightsquigarrow$$

$\mathfrak{g}[t,t^{-1}] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$ with Lie bracket

- $[x\otimes t^n, y\otimes t^m] = [x,y]\otimes t^{n+m}$.

Consider 2-cocycle: $\boxed{\omega(F,G) = \mathrm{Res}_{t=0}\,(G, dF) \quad \forall F,G \in \mathfrak{g}[t,t^{-1}]}$

or more explicitly

$$\boxed{\omega(\underset{\mathfrak{g}}{x}\otimes t^m, \underset{\mathfrak{g}}{y}\otimes t^n) = \underbrace{(x,y)\,\delta_{m,-n}\cdot m}_{(x,y)\cdot \mathrm{Res}_{t=0}\, t^n dt^m}}$$

Exercise: Check it is a 2-cocycle!

Recall : $\mathfrak{g}$ - simple if it's not abelian and has no proper ideals.

($J \subseteq \mathfrak{g}$ - ideal if $[J, \mathfrak{g}] \subseteq J$)

$\boxed{\text{Fact}}$: If $\mathfrak{g}$ - simple $\Rightarrow$ ~~all~~ the space of invariant forms is 1-dim, with explicit generator, Killing form,

$$(x, y) = \mathrm{Tr}_{\mathfrak{g}}\, \mathrm{ad}(x)\mathrm{ad}(y)$$

$\mathrm{ad}(x) := [x, -]$.
linear operator $\mathfrak{g} \to \mathfrak{g}$

Basic example

$\mathfrak{g} = \mathfrak{sl}_n$ = traceless $n \times n$ matrices

$(X, Y) = \mathrm{Tr}(X \cdot Y)$

Thm 2 : If $\mathfrak{g}$ - simple, then $H^2(\mathfrak{g}[t, t^{-1}])$ - 1-dim, spanned by aforementioned $\underline{\omega}$!

affine (untwisted) Kac-Moody alg.

$$\Rightarrow \boxed{\hat{\mathfrak{g}} := \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} \cdot K}$$

central

with $[\,,\,]$: $\boxed{[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + (x, y) \cdot n\, \delta_{m,-n} K}$

Proof of Thm 2 relies on one classical result on simple Lie alg-s.

- $\mathfrak{g}$ - Lie alg., M - $\mathfrak{g}$-module.

$Z^1(\mathfrak{g}, M) = \{\varphi : \mathfrak{g} \to M \mid \varphi([x,y]) = x(\varphi(y)) - y(\varphi(x))\}$

$\cup$

$B^1(\mathfrak{g}, M) = \{\varphi : \mathfrak{g} \to M \mid \exists m \in M \text{ s.t. } \varphi(x) = x(m)\}$

$H^1(\mathfrak{g}, M) = \dfrac{Z^1(\mathfrak{g}, M)}{B^1(\mathfrak{g}, M)}$ $\leftarrow$ 1st cohom. of $\mathfrak{g}$ w/ coeff in M.

Whitehead Lemma: If $\mathfrak{g}$ - simple, M - f. dim $\Rightarrow$ $\boxed{H^1(\mathfrak{g}, M) = 0}$

! Apply this to the case when you have ambient Lie alg. L,

$\mathfrak{g} = I \subseteq L$ - Lie subalg, $M \subset L$ - I-module.

Then: if $\omega \in Z^2(L) \rightsquigarrow \varphi \in Z^1(L, M^*)$ $\leftarrow$ Hwk Exercise $\varphi \in Z^1$!

$\omega \longmapsto \varphi(x)(m) = \omega(x, m)$.

$$L \overset{\rho}{\curvearrowright} M$$

$$\begin{array}{ccc} L & \overset{\rho^*}{\curvearrowright} & M^* \\ \psi & & \psi \\ x & & \xi \end{array}$$

dual of M

$$\boxed{(\rho^*(x)\xi)(m) = -\xi(\rho(x)m)}$$

must add "−"!

(a shadow of its group version
where $(\rho^*(g)\xi)(m) := \xi(\rho(g^{-1})m)$)

Dual module over Lie algebra

<u>Proof of Thm 2</u>

$L = \mathfrak{g}[t,t^{-1}]$, $\quad \underset{\underset{\mathfrak{g}}{\wr\wr}}{I} = \mathfrak{g}\otimes t^0 \subset L$, $\quad \underset{\underset{I}{\circlearrowleft}}{M} = \mathfrak{g}\otimes t^n \subset L$ (in the above setup p.7)

<u>By Whitehead Lemma</u>: $\underbrace{H^1(\mathfrak{g},\mathfrak{g}^*)=0}$

$H^1(I, M^*)=0$

- Back to thm. Let's pick <u>any</u> $\beta \in Z^2(\mathfrak{g}[t,t^{-1}])$
- $\beta|_{I\times M} \rightsquigarrow$ produces elt of $Z^1(I,M^*) = \underline{B^1(I,M^*)}$

$\Rightarrow \exists\, m^* \in M^*$ s.t. $\beta(x, y\cdot t^n) = -m^*([x, y\cdot t^n])$

<u>Therefore</u>: can modify β by $\underbrace{\text{2-coboundary}}_{\text{elt of } B^2(\mathfrak{g}[t,t^{-1}])}$

$\xi: \mathfrak{g}[t,t^{-1}] \to \mathbb{C}$

$\xi|_{\mathfrak{g}t^n} = -m^*$

<u>Upshot</u>: Changing by 2-coboundary $\beta \rightsquigarrow \tilde{\beta}$

s.t. $\boxed{\tilde{\beta}(x, yt^n) = 0 \quad \forall n \quad \forall x, y}$

<u>Step 2</u>: Apply 2-cocycle cond-n on $\tilde{\beta}$ with

$a = x, \quad b = y \cdot t^n, \quad c = z \cdot t^m. \qquad n, m \in \mathbb{Z}.$

$$\underbrace{\tilde{\beta}([x,y]t^n, z \cdot t^m)}_{} + \underbrace{\tilde{\beta}([y,z]t^{n+m}, x)}_{=0 \text{ by step1}} + \underbrace{\tilde{\beta}([z,x]t^m, yt^n)}_{} = 0.$$

$$\boxed{\tilde{\beta}([x,y]t^n, z \cdot t^m]) + \quad 0 \quad + \tilde{\beta}(yt^n, [x,z]t^m) = 0}$$

<u>That means</u>: bilinear map $\boxed{\tilde{\beta}_{m,n}(a, b) := \tilde{\beta}(at^n, bt^m)}$ is inv. form on $\mathfrak{g}$!

($a, b \in \mathfrak{g}$)

Since all invariant forms are proportional to each other (as $\mathfrak{g}$-simple) $\Rightarrow$ $\tilde{\beta}_{m,n} = \boxed{\lambda_{m,n}} \cdot \underbrace{(\,,\,)}_{\text{choice from beginning}}$

constants

We shall get constraints on $\lambda_{m,n}$ in the next step

Final Step: Apply 2-cocycle cond-n to general case: $\begin{cases} a = xt^n \\ b = yt^m \\ c = zt^p \end{cases}$

$\Downarrow$

$$\boxed{([x,y],z)\cdot \lambda_{n+m,p} + ([y,z],x)\cdot \lambda_{m+p,n} + ([z,x],y)\cdot \lambda_{p+n,m} = 0.}$$

$([x,y],z) = -(y,[x,z]) = ([z,x],y) \dots = ([y,z],x)$ $\Downarrow$

$$(*) \quad \boxed{\lambda_{n+m,p} + \lambda_{m+p,n} + \lambda_{p+n,m} = 0} \quad \forall m,n,p.$$

$n=m=0 \Rightarrow \lambda_{0,p} = 0 \ \forall p.$

$m=1, \ p = N-n-1 \Rightarrow \lambda_{n,N-n} + \lambda_{1,N-1} = \lambda_{n+1,N-n-1}$ $\Big\} \rightarrow \boxed{\lambda_{n,N-n} = n\cdot\lambda_{1,N-1}}$

$m=-n \Rightarrow \boxed{\lambda_{n,-n} = n\cdot\lambda_{1,-1}}$ $\quad$ $\boxed{\textbf{Also}: \lambda_{m,n} = 0 \text{ if } m+n \neq 0}$

← Verify this at home! ↗

Lemma 1 (Dixmier's lemma = Schur Lemma in countable dim-n)

Let V be a countably dimensional irred. repr. of an assoc. alg. $A/\mathbb{C}$. Then any endom. of V, commuting with A, is scalar:

$$\boxed{\mathrm{End}_A(V) = \mathbb{C}\cdot \mathrm{Id}_V.}$$

! In particular, if A itself is countably dim-l ~~...~~ (then necessarily V is $\leq$ countably dim-l).

$\Downarrow$

$$\boxed{\mathrm{End}_A(V) = \mathbb{C}\cdot \mathrm{Id}_V.}$$

Proof:

Contradiction: ϕ - not scalar.

- $\underbrace{\phi \in \mathrm{End}_A(V)}_{D=}$ $\qquad V \xrightarrow{\phi} V$

division algebra (V-irr $\Rightarrow$ $\mathrm{Im}\,\phi = V$, $\mathrm{Ker}\,\phi = 0$ $\Rightarrow$ ϕ - invertible)

$\Rightarrow \nexists\, P$ - pol-l with $\mathbb{C}$-coeff. s.t. $\underbrace{P(\phi)=0}$ (Factor P and use $\phi \neq$ scalar) $\Rightarrow$ ϕ - transcendental/$\mathbb{C}$. $\Rightarrow \boxed{\mathbb{C}(\phi) \subseteq D}$

- D - countably dimensional.

Pick any $v \in V \Rightarrow \phi$ is uniquely determined by $\phi(v) \in \underline{V}$. (countable dim-l)

$A(v) = V$

$\Rightarrow \boxed{D \text{ - countably dim-l}}$

not countable (as parametrized by $\mathbb{C}$).

$\underbrace{\left\{\frac{1}{\phi - a}\right\}_{a\in\mathbb{C}}} \subseteq D$ (countable) $\Rightarrow$ ↯. $\Rightarrow \boxed{\phi \text{ - scalar}}$

- $\phi - a \in \mathrm{End}_A(V)$ is nonzero as $\phi \neq$ scalar $\Rightarrow$ admits inverse (as D-division alg) $\Rightarrow$ notation $\frac{1}{\phi - a}$
- If some $\frac{1}{\phi - a}$ were lin. dep. $\Rightarrow$ • common denominator to get $P(\phi) = 0$ with P-pol-l $\Rightarrow$ ↯.

Corollary: In the above setting, if $c \in A$ - central, then $c|_V$ is a scalar operator (i.e. $\lambda \cdot \mathrm{Id}_V$)

irreducible.

$$A \overset{\rho}{\curvearrowright} V \qquad ca = ac \quad \forall a \in A$$

$$A \ni c \qquad \rho(c)\rho(a) = \rho(a)\rho(c) \Rightarrow \boxed{\rho(c) \in \mathrm{End}_A(V)}$$

$\Downarrow$ $\rho(c) = \lambda \,\mathrm{Id}_V$ by Lemma ∎

Next time: Apply this to $A = \mathcal{U}(\mathcal{A})$

univ. enveloping alg.

oscillator algebra

After that: general framework of $\mathbb{Z}$-graded Lie alg-s.