Lecture #3

01/26/2021

Last time:
- $H^2(W)$ - 1-dim
- $H^2(\underbrace{\mathfrak{g}[t,t^{-1}]})$ - 1-dim if $\mathfrak{g}$-simple f.d.

$$\boxed{\omega(xt^m, yt^n) = (x,y)\cdot\delta_m^{-n}\cdot m}$$

$x,y\in\mathfrak{g}$
$m,n\in\mathbb{Z}$

$\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}\cdot K$ ← (untwisted) affine Kac-Moody alg.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\uparrow$ central

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Dixmier's lemma (= update of Schur lemma).

$\Downarrow$

Cor: If A is an assoc. alg./$\mathbb{C}$ of countable dimension, V- irreducible repr. of A $(\rho: A\to\mathrm{End}(V))$ $\Big\}\Rightarrow$ $\boxed{\underbrace{\mathrm{End}_A}\, V = \mathbb{C}}$ $\wr\downarrow$ λ's

all linear operators $\phi: V\to V$ s.t. $\phi\rho(a)=\rho(a)\phi$ $\forall a\in A$

$\lambda\cdot\mathrm{Id}_V$ ← identity operator

$\Downarrow$

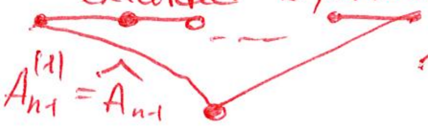
Cor: If $c\in A$ is central $\Rightarrow$ it acts by scalars on all irred. reps.
as above

Rmk: $\hat{\mathfrak{g}}$ has a completely different realization; $\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K$ $\overset{\text{THM}}{\|}$

e.g. if $\underline{\underline{\mathfrak{g}=\mathfrak{sl}_n}}$

A_{n-1} $\rightsquigarrow$

extended Dynkin diagr.



$A^{(1)}_{n-1}=\hat{A}_{n-1}$ $\rightsquigarrow$ $\mathfrak{g}_{KM}=\hat{A}_{n-1}$

$\mathcal{A}$ = oscillator algebra

↑ generators: $\{a_n\}_{n\in\mathbb{Z}} \cup \{\overset{\text{central}}{K}\}$

$$\boxed{[a_n, a_m] = n\,\delta_n^{-m}\cdot K}$$

$\mathcal{A}$ - repr-s (under mild constraints)

Rmk: If $\mathfrak{g}$ - Lie alg., then a repr-n of $\mathfrak{g}$ is a vector space V together with $\overset{\text{linear}}{\rho}: \mathfrak{g} \longrightarrow \mathrm{End}(V)$ s.t. $\rho([x,y]) = \underbrace{[\rho(x), \rho(y)]}_{\rho(x)\rho(y) - \rho(y)\rho(x)}$

Lie alg ⟷ Assoc. algebra

$\underset{\text{}}{\mathfrak{g}} \longleftrightarrow \mathcal{U}(\mathfrak{g})$ - universal enveloping algebra

$$\underbrace{\mathcal{U}(\mathfrak{g})} \parallel \underbrace{T(\mathfrak{g})}/(x\otimes y - y\otimes x - [x,y])$$

↗

$$\underline{\text{Tensor alg}} = \mathbb{C} \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}^{\otimes 3} \oplus \dots$$

$$\underbrace{(x\otimes y)}_{\in\,\mathfrak{g}^{\otimes 2}} \cdot \underbrace{(z\otimes u\otimes v)}_{\in\,\mathfrak{g}^{\otimes 3}} \in \mathfrak{g}^{\otimes 5}$$

Exercise: $\underset{\nwarrow \text{Lie alg}}{\mathfrak{g}}$ - modules/repr-s $\longleftrightarrow$ $\underset{\text{assoc. alg.}}{\underline{\mathcal{U}(\mathfrak{g})}}$ - modules/representations

Important result about $\mathcal{U}(\mathfrak{g})$ — the PBW theorem!

If $\{X_i\}_{\underset{\uparrow\text{totally ordered set}}{i\in I}}$ – basis of $\mathfrak{g}$, then $\underbrace{\{X_{i_1}\otimes X_{i_2}\otimes\dots\otimes X_{i_k}\}^{k\geq 0}_{i_1\leq i_2\leq\dots\leq i_k}}_{\uparrow\text{is a basis of }\mathcal{U}(\mathfrak{g})}$

(i.e. $\mathcal{U}\mathfrak{g}$ has roughly the same size as the symmetric algebra $S(\mathfrak{g})$)

$\overset{\swarrow\text{oscillator/Heisenberg Lie alg.}}{A}$ – repr-s $\longleftrightarrow$ $\underline{\overset{\swarrow\text{countable dimensional.}}{\mathcal{U}(A)}\text{-repr-s}}$ $\Rightarrow$ can apply Dixmier's lemma & its corollaries

K – central el-t of $\mathfrak{g}$ $\Rightarrow$ also of $\mathcal{U}(A)$ $\Rightarrow$ K acts by scalar ($k\cdot\mathrm{Id}_V$) on all irred. reps.

Case 1 ($k=0$, i.e. $K|_V=0$) $[a_n,a_m]=n\delta_n^{-m}\cdot K$ $\Rightarrow$ $\{a_n\}$ acting on V by pairwise commuting operators.

$\Rightarrow$ all are central $\xRightarrow{\text{Cor2}}$ $\overset{\forall n}{a_n}$ acts by scalars on any irreducible repr-n of A (on which K acts via 0).

V-irred ⇒ V- 1-dim ⇒ <u>very simple</u>!

<u>Case 2</u> ($k \neq 0$) WLOG can assume that $\boxed{k=1}$

(b/c there is ~~an~~ an $\overset{\text{Lie}}{\text{alg.}}$ ~~homom.~~ isomorphism $A \to A$
which <u>rescales</u> the value of k.
$K \mapsto k \cdot K$
$a_i \mapsto a_i$ if $i \leq 0$
$a_i \mapsto k a_i$, if $i > 0$.)

$$\overset{\text{Lie alg}}{A} \curvearrowright V \longleftrightarrow \mathcal{U}(A) \curvearrowright V \longrightarrow \boxed{\mathcal{U}(A)/(K-1) \curvearrowright V}$$

<u>Prop 1</u>: The assignment ($j>0$)

$$\varphi: a_{-j} \mapsto x_j, \quad a_j \mapsto j\frac{\partial}{\partial x_j}, \quad K \mapsto 1, \quad a_0 \mapsto x_0 \qquad (1)$$

gives rise to an algebra isomorphism

$$\boxed{\mathcal{U}(A)/(K-1) \xrightarrow{\sim} \underbrace{\mathcal{D}iff(x_1, x_2, \ldots)}_{\text{Diff-l operators in } \{x_i\}_{i>0} \text{ with pol-l coeff-s.}} \otimes \mathbb{C}[x_0]}$$

$a_m a_n - a_n a_m - m\delta_m^{-n} K = 0.$

$[j\frac{\partial}{\partial x_j}, x_j] = j$ ⇗

Step 1: F-las (1) indeed give rise to a homom LHS → RHS.

Step 2 : Show it's both injective & surjective

Surj : RHS body generated by $\{x_j\}_{j>0}$, $\{\frac{\partial}{\partial x_j}\}_{j>0}$, $\{x_0\}$; all in the image!

Inj : Use PBW to argue that

$$\left.\begin{array}{l} \underbrace{\prod_{j>0} a_{-j}^{?} \cdot \prod_{j>0} a_j^{?} \cdot a_0^{?}}_{\Big\downarrow \text{ obvious basis of RHS}} \text{ — a basis of } \dfrac{\underline{LHS}}{\mathcal{U}(A)/(K-1)} \end{array}\right\} \Rightarrow \text{Done!}$$

Corollary 3 (Definition of Fock modules).

$$\underbrace{\mathrm{Diff}(x_1, x_2, \dots) \otimes \mathbb{C}[x_0]}_{} \curvearrowright \boxed{\mathbb{C}[x_1, x_2, x_3, \dots] = F_\mu}$$

$$x_0 \longmapsto \mu \cdot \text{Identity}$$

$\|$ differential operator $\longmapsto$ acts naturally.

$\mathcal{U}(\mathcal{A})/(K-1)$

$$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}$$

Get a natural action $\mathcal{U}(\mathcal{A})/(K-1) \curvearrowright F_\mu \Rightarrow \boxed{\mathcal{A} \curvearrowright F_\mu}$

$\begin{cases} K \text{ acts by Id} \\ a_0 \text{ acts by } \mu\,\text{Id} \end{cases}$

(**Rmk**: Both K, a_0 are central in $\mathcal{A}$)

Prop 2: $\mathcal{A}$-Modules $\{F_\mu\}^{\mu \in \mathbb{C}}$ are irreducible and pairwise non-isomorphic.

- $a_0|_{F_\mu} = \mu\cdot Id \Rightarrow F_\mu \not\cong F_\nu$ if $\mu \neq \nu$.

- Let's assume the contrary, i.e. ∃ subrepr-n $\overset{\circlearrowleft \mathcal{A}}{V} \subseteq \underset{\shortparallel\ \mathbb{C}[x_1, x_2, \ldots]}{F_\mu}^{\circlearrowleft \mathcal{A}.}$

Take any $\overset{\overset{0}{\neq}}{v} \in V$, i.e. $v \in \mathbb{C}[x_1, x_2, \ldots]$

$$\Downarrow$$

∃ Diff. operator $\mathcal{D}$ s.t. $\boxed{\mathcal{D}(v) = 1 \in} V$

$\mathcal{U}(\mathcal{A})\Big/\left(\begin{smallmatrix}K-1\\ a_0-\mu\end{smallmatrix}\right) \xrightarrow{\sim} \mathcal{D}iff(x_1, x_2, \ldots) \Rightarrow \mathcal{D}$ is in the image of $\mathcal{U}(\mathcal{A})$

$\Rightarrow \exists\, a \in \mathcal{U}(\mathcal{A}) : a(v) = 1$.

applying further $\{a_{-j} \mid j > 0\}$ get the entire F_μ.

$$\Downarrow\quad F_\mu \subseteq V \Rightarrow V = F_\mu.\ \boxtimes$$

Q: a) Are those all irred. repr-s of $\mathcal{U}(A)/(K-1)$?

b) What can we say about non-irred. repr-s?

Exercise: Construct irreducible repr-s of $\mathcal{U}(A)/(K-1)$ which are not isom. to F_μ.

Hint: Consider the action on $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots] \circ \underbrace{x_1^{\lambda_1} x_2^{\lambda_2} \cdots}_{\lambda_1, \lambda_2, \dots \notin \mathbb{Z}}$.

<u>Prop3</u>: a) Let V be an irreducible A-module in which $K \mapsto 1$, $a_0 \mapsto \mu$, and s.t.

[$\forall v \in V$: $\underbrace{(\mathbb{C}[a_1, a_2, \dots] v)}$ - fin.dimensional and moreover all $\{a_i\}_{i>0}$ act in this space by nilpotent operators

$$\Rightarrow \underline{V \simeq F_{\mu}},$$

b) Let V be an A-module (**not** necessarily irreducible) as in (a) and <u>also</u> s.t.

$$\boxed{\forall v \in V \ \exists N \text{ s.t. } a_{>N}(v) = 0}$$

$$\Longrightarrow V \simeq F_{\mu} \oplus F_{\mu} \oplus \dots \simeq F_{\mu} \otimes M$$

↑ multiplicity vector space.

a) Pick $v \in V\setminus\{0\} \rightsquigarrow \underbrace{\mathbb{C}[a_1, a_2, \dots]v}$

$\|_{\rho}$

W — fin. dim. subspace

$\circlearrowleft \{a_i\}_{i>0}$ act nilpotently

$[a_i, a_j] = 0$ if $i, j > 0$.

$\Big\} \longrightarrow$ they have a common eigenvector with eigenvalue **ZERO**

$\Rightarrow \exists\, w \in W \subset V\setminus\{0\}$ s.t. $\boxed{\begin{array}{l} a_i(w) = 0 \\ \forall i > 0 \end{array}}$

$\Rightarrow$ there is a homomorphism of $\mathcal{A}$-repr-s

$$\boxed{\begin{array}{ccc} F_\mu & \xrightarrow{\text{nonzero}} & V \\ 1 & \longmapsto & w \end{array}}$$

irred $\to F_\mu$; $V \leftarrow$ irred.

$F_\mu = \mathbb{C}[x_1, x_2, x_3, \dots]$

$\circlearrowleft$

$\mathcal{A}$ acts via

$K \mapsto \mathrm{Id}$

$a_0 \mapsto \mu \cdot \mathrm{Id}$

$a_{-j} \mapsto x_j$

$a_{+j} \mapsto j\frac{\partial}{\partial x_j}$.

V

$\rotatebox{90}{$\in$}$

w

$\rightsquigarrow$

$\mathcal{U}(\mathcal{A}) \longrightarrow V$

$x \longmapsto \rho(x)w$

Sketch: $F_\mu \xleftarrow{\sim} \mathcal{U}(\mathcal{A}) \Big/ \left(\begin{array}{l} K-1 \\ a_0 - \mu \\ a_j, j > 0 \end{array} \right)$

Schur Lemma $\Rightarrow F_\mu \simeq V$. $\Rightarrow$ proves (a).

(b) We will use a trick involving the Euler field:

$$\boxed{E = a_{-1}a_1 + a_{-2}a_2 + \dots = \sum_{j>0} a_{-j} a_j \in \widehat{\mathcal{U}(\mathcal{A})}}$$ ← completion of $\mathcal{U}(\mathcal{A})$ (as it is infinite sum)

Due to our assumption: $\forall v \in V$ $a_j(v) = 0$ for $j \gg 1$

$\Rightarrow$ $\boxed{\text{E defines an "honest" linear operator on V.}}$!

$v \in V \setminus \{0\} \rightsquigarrow \underbrace{M := \mathbb{C}[a_1, a_2, \dots] v}_{\text{f. dim subspace}}$

$\mathbb{C}[a_1, a_2, \dots] \longrightarrow M$ ← Let I_v be the kernel of that map.

$P(a_1, a_2, \dots) \longmapsto P(a_1, a_2, \dots) v$

Know: $\dim\left(\mathbb{C}[a_1, a_2, \dots] / I_v\right) < \infty$.

A-submodule

Now: Let $W := \mathcal{U}(\mathcal{A}) v \subset V$.

$\mathrm{Diff}(x_1, x_2, \dots) \twoheadrightarrow W$

$\mathrm{Diff}(x_1, x_2, \dots) \circ I_v \subseteq \mathrm{Kernel}(\quad)$

b/c $\mathcal{U}(\mathcal{A}) / (K-1) \xrightarrow{\sim} \mathrm{Diff}(x_1, x_2, \dots) \otimes \mathbb{C}[x_0$

$\Downarrow$

$\mathcal{U}(\mathcal{A}) / \left(\begin{smallmatrix} K-1 \\ a_0 - \mu \end{smallmatrix}\right) \xrightarrow{\sim} \mathrm{Diff}(x_1, x_2, \dots)$

Exercise (next hwk *): The quotient

$\widetilde{W} := \mathrm{Diff}(x_1, x_2, \text{—})/\mathrm{Diff}(x_1, x_2, \text{—})\cdot I_v$ is a finite-length module of $A/(K-1)$ with all composition factors $\simeq F_\mu$ (and the length is $=$ $\dim M = \dim(\mathbb{C}[x_1, x_2, \dots]/I_v)$)

(Recall: $A \curvearrowright V$

$0 \subset \underbrace{V_1 \subset V_2 \subset \dots \subset V_{k-1}}_{\text{submodules}} \subset V_k = V$

V_{i+1}/V_i - irreducible)

see Jordan - Holder theorem

BUT: $\widetilde{W} \twoheadrightarrow W$ by above

$\Longrightarrow$ $\boxed{\text{W is also of finite length with all consequent quotients} \simeq F_\mu.}$

V

$\cup$

$\forall v \rightsquigarrow W = \mathcal{U}(A)v$

$\uparrow$ submodule containing v

$$E = \sum_{j>0} a_{-j} a_j \curvearrowright V$$

When $V = F_\mu$, $E = \underbrace{\sum_{j>0} j x_j \frac{\partial}{\partial x_j}} \curvearrowright F_\mu = \mathbb{C}[x_1, x_2, \dots]$

all eigenvalues are $\in \mathbb{Z}_{\geq 0}$!

$V \supset W \circlearrowleft E$ ↙ finite length filtration by A-submodules

$0 \subset W_1 \subset W_2 \subset \dots \subset W_k = W$

all quotients are $\simeq F_\mu \circlearrowleft E$

(follows from $E \curvearrowright V$ loc. finitely, i.e. $\forall v \in V$ $\mathbb{C}[E]v$ - f.dim)

$\Downarrow$

$$\boxed{V = \bigoplus_{k \geq 0} V[k] \text{ — generalized eigenspaces for the action of } E}$$

$$\boxed{V = \bigoplus_{k \geq 0} V[k].}$$

Obvious: if $v \in V$ s.t. $a_{>0}(v) = 0 \Rightarrow E(v) = 0. \Rightarrow E^N(v) = 0 \ \forall N > 0.$

! **BUT**: the opposite is true; i.e. $E^N(v) = 0 \Rightarrow a_j(v) = 0 \ \forall j > 0$

Assume the contrary, i.e. $\exists j > 0$ s.t. $a_j(v) \neq 0$.

$$\underbrace{[E, a_j] = -j a_j}_{\checkmark} \longleftarrow \text{follows from } [a_{-j}, a_j] = -j \cdot K, \ \overset{i \neq j}{[a_{-i}, a_j] = 0}$$

$$(E + j)^s a_j(v) = \underbrace{a_j \cdot E^s(v)}_{\overset{\|}{0} \text{ for } s \gg 1} \quad \forall s \quad \Rightarrow \quad a_j(v) \in V[-j]$$

$\Downarrow$ as all generalized eigenvalues are in $\mathbb{Z}_{\geq 0}$. ↯

generalized eigenvector w/ eigenvalue $-j$.

So:

$$\Rightarrow \boxed{V[0] = \left\{ \begin{matrix} \text{Eigenspace for } E \\ \text{with eigenvalue} = 0 \end{matrix} \right\} = \{ v \mid a_{>0}(v) = 0 \}.}$$

Defined as the generalized eigenspace for E with eigenvalue $=0$.

$$V \xleftarrow{\text{repr. homom.}} \underbrace{V[0]}_{\text{multiplicity space}} \otimes \overset{\text{Fock}}{F_\mu}$$

(easy) Exercise: it is injective.

$\Rightarrow$ Thus, it suffices to show it's surjective

$$\begin{tikzcd} F_\mu \otimes \underbrace{V[0]}_{} \arrow[r, hook] & V \arrow[d, squiggly] \\ & \left(V/F_\mu \otimes V[0]\right) \circlearrowleft A \end{tikzcd}$$

Repeat above arguments for this quotient module instead of V.

!!! must be empty by construct.

$\underbrace{\left(V/F_\mu \otimes V[0]\right)}$ look at generalized eigenspace of Euler field E with eigenvalue 0

But on the other hand

Take any nonzero $v \rightsquigarrow W$ – has finite filt.

$0 \subset W_1 \subset W_2 \subset \dots \subset W_k = W$

$W_1 \simeq F_\mu \circlearrowleft E$ has eigenvalue 0

above product

$\Downarrow$

there are actually eigenvectors with eigenv. 0.

Thus we obtain a contradiction with $\left(V/F_\mu \otimes V[0]\right)[0] = 0$.