Lecture 4

01/28/2021

Last time: Dixmier's Lemma $\Rightarrow$ classify all (irred) repr-s of oscillator algebra $\mathcal{A}$ under some assumptions

$\hookrightarrow$ key ingredient from the end: Euler field

$$E = \sum_{i>0} a_{-i} a_i \in \mathcal{U}(\mathcal{A})^{\wedge} \quad \leftarrow \text{completion}$$

via $\sum i x_i \frac{\partial}{\partial x_i}$ on F_μ.

! BUT: E acts on all repr-s satisfying above assumptions

Rmk: $\mathcal{A}$ has a basis $\{a_n\}_{n\in\mathbb{Z}} \cup \{K\}$.

$\mathbb{Z}$-graded algebra with $\deg(a_n)=n$, $\deg(K)=0$

$[K, a_n] = 0 \qquad [a_m, a_n] = m\delta_{m,-n} \cdot K$

Fock module

$\mathcal{A} \curvearrowright F_\mu = \mathbb{C}[x_1, x_2, x_3, \dots] \quad \leftarrow$ $\mathbb{Z}$-graded module.

$\deg(x_i) = -i$

$F_\mu = \bigoplus_{n\geq 0} F_\mu[-n] \qquad F_\mu[0] = \mathbb{C} \qquad F_\mu[-1] = \mathbb{C}\cdot x_1 \qquad F_\mu[-2] = \mathbb{C}\cdot x_1^2 + \mathbb{C}\cdot x_2 \dots$

Character:
$$\operatorname{Tr}_{F_\mu}(q^E) := \sum_{n\geq 0} \dim(F_\mu[-n])\, q^n = \sum_{n\geq 0} p(n)\cdot q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots}$$

Today : Z-graded Lie algebras

Def : Lie alg. $\mathfrak{g}$ is Z-graded if it is equipped with a v. space decomp.

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n \text{ s.t. } [\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n} \quad \forall m,n \in \mathbb{Z}$$

Examples : $\mathcal{A}$, $\deg a_n = n$, $\deg K = 0$

W, $\deg L_n = n$

Vir, $\deg L_n = n$, $\deg C = 0$.

To get some uniform repr. theory results, we'll treat particular class of Z-graded Lie algebras, as defined in the next definition:

Def : A Z-graded Lie alg. $\mathfrak{g} = \bigoplus \mathfrak{g}_n$ is nondegenerate if:

1) $\dim \mathfrak{g}_n < \infty$ $\forall n$

2) $\dim \mathfrak{g}_0$ - abelian

3) $\forall n > 0$ and generic $\lambda \in \mathfrak{g}_0^*$ (Zariski topol.), the pairing $\mathfrak{g}_n \times \mathfrak{g}_{-n} \longrightarrow \mathbb{C}$ - nondegenerate

$$(x, y) \longmapsto \lambda(\underbrace{[x,y]}_{\in\,\mathfrak{g}_0})$$

(Note: in particular, $\dim \mathfrak{g}_n = \dim \mathfrak{g}_{-n}$ $\forall n$.)

Ex: 1) A, $\deg a_n = n$, $\deg K = 0$. $\Rightarrow \begin{cases} A_n = \mathbb{C}\cdot a_n, & n \neq 0. \\ A_0 = \mathbb{C}\cdot a_0 \oplus \mathbb{C}\cdot K \end{cases}$

$\underset{\text{1-dim}}{A_n} \times \underset{\text{1-dim}}{A_{-n}} \longrightarrow \mathbb{C}$ - obviously non-deg. for generic λ.

2) Witt alg, Vir.

$Vir_n = \mathbb{C}\cdot L_n$, $n \neq 0$

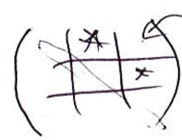
$Vir_0 = \mathbb{C}\cdot L_0 + \mathbb{C}\cdot K$

non-degeneracy is obvious. (as all components are 1-dim, besides for Vir_0)

3) $\mathfrak{g}$ - simple Lie algebra $\leftarrow$ $\mathfrak{g}$ is gen-d by $\{e_i, f_i, h_i\}$ $\swarrow$ Chevalley generators

"principal grading": $\deg(e_i) = 1$, $\deg(f_i) = -1$, $\deg(h_i) = 0$.

Ex: $\mathfrak{g} = \mathfrak{sl}_3$



$\mathfrak{g}_1 = \mathbb{C}\cdot E_{12} + \mathbb{C}\cdot E_{23}$ $\quad$ $\mathfrak{g}_{-1} = \mathbb{C}\cdot E_{21} + \mathbb{C}\cdot E_{32}$

$\mathfrak{g}_2 = \mathbb{C}\cdot E_{13}$

$\mathfrak{g}_{-2} = \mathbb{C}\cdot E_{31}$.

! **Exercise**: non-degenerate! (Find a basis of $\mathfrak{g}_n$, $\mathfrak{g}_{-n}$ which are dual; use root vectors of $\mathfrak{g}$ $\longrightarrow$)

4*) For affine (untwisted) Kac-Moody: $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\cdot K$.

"Principal grading": $\deg(f_\theta \cdot t) = 1$, $\deg(e_\theta \cdot t^{-1}) = -1$ $\quad$ θ - highest root

above example of $\mathfrak{g} = \mathfrak{sl}_3$: $\deg\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} = 1 = -\deg\begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Motivation: develop uniform repr. theory — including $\mathcal{A}$, Vir, $\hat{\mathfrak{g}}$.

Observation: $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ $\qquad$ $\mathfrak{g}_0$ – abelian.

Set: $\boxed{\mathfrak{n}_- := \bigoplus_{n<0} \mathfrak{g}_n, \quad \mathfrak{n}_+ := \bigoplus_{n>0} \mathfrak{g}_n, \quad \mathfrak{h} = \mathfrak{g}_0.}$

$\Downarrow$

$$\boxed{\mathfrak{g} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+} \quad \leftarrow \text{as vector spaces}$$

triangular decomposition.

$\Downarrow$

$$\boxed{\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}_+)}$$

Key Def: a) For $\lambda \in \mathfrak{h}^* (= \mathfrak{g}_0^*)$, the *highest-weight Verma module* $M_\lambda^+ = M_\lambda$ over $\mathfrak{g}$ is defined as:

$$M_\lambda^+ := \mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda \overset{\text{def}}{=} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda .$$

where $\mathbb{C}_\lambda \ni 1_\lambda$ is a 1-dim repr-n of $\mathfrak{h} \oplus \mathfrak{n}_+$ ($\mathfrak{n}_+$ acts by zero, $\mathfrak{h}$ acts via λ):

$x \in \mathfrak{n}_+ : \ x(1_\lambda) = 0$
$x \in \mathfrak{h} : \ x(1_\lambda) = \lambda(x) \cdot 1_\lambda$

b) Likewise, the lowest-weight Verma module is

$$M_\lambda^- := \mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}} \mathbb{C}_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_-)} \mathbb{C}_\lambda .$$

($\mathfrak{n}_-$ acts trivially on $\mathbb{C}_\lambda$.)

Lemma 1: a) $M_\lambda^\pm$ are $\mathbb{Z}$-graded $\mathfrak{g}$-modules

(i.e. $M_\lambda^\pm = \bigoplus M_\lambda^\pm[n]$, s.t. $x(m) \in M_\lambda^\pm[m+n]$ for $x \in \mathfrak{g}_m$, $m \in M_\lambda^\pm[n]$)

b) $M_\lambda^+ \xleftarrow{\sim} \mathcal{U}(\mathfrak{n}_-)$ (isom. of $\mathbb{Z}$-graded v. spaces), $x(v_\lambda^+) \longleftarrow\!\shortmid x$,

$M_\lambda^- \xleftarrow{\sim} \mathcal{U}(\mathfrak{n}_+)$ (isom. of $\mathbb{Z}$-graded v. spaces), $M_\lambda^- \ni x(v_\lambda^-) \longleftarrow\!\shortmid x \in \mathcal{U}(\mathfrak{n}_+)$

where $v_\lambda^\pm$ are images of $1_\lambda \in \mathbb{C}_\lambda$ ← the highest/lowest weight vectors of $M_\lambda^\pm$.

a) $\mathfrak{g}$ - $\mathbb{Z}$-graded Lie alg $\Rightarrow$ $\mathcal{U}(\mathfrak{g})$ - $\mathbb{Z}$-graded. $\supseteq \mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_+)$ - $\mathbb{Z}$-graded

$M_\lambda^+ = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$, as $\mathfrak{n}_+ \curvearrowright \mathbb{C}_\lambda$ trivially

$\mathbb{C}_\lambda$ is a $\mathbb{Z}$-graded module $\mathfrak{h}\oplus\mathfrak{n}_+$.

$\Rightarrow M_\lambda^+$ is obviously $\mathbb{Z}$-graded repr-n.

b) PBW thm $\Rightarrow$ v. space isom. $\mathcal{U}(\mathfrak{n}_-)\otimes\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_+) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$.

$M_\lambda^+ = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda \overset{\text{v. space}}{\simeq} \mathcal{U}(\mathfrak{n}_-)$ (easy: compatible with $\mathbb{Z}$-gradings)

So: $M_\lambda^\pm$ - $\mathbb{Z}$-graded modules of $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$.

$\Downarrow$

$M_\lambda^+ = \bigoplus_{n\geq 0} M_\lambda^+[-n]$, $M_\lambda^+[-n] = \mathcal{U}(\mathfrak{n}_-)[-n]\, v_\lambda^+$ (so that $\mathcal{U}(\mathfrak{n}_-)[-n] \xrightarrow{\sim} M_\lambda^+[-n]$ as v. spaces)

Character fla:
$$\sum_{n\geq 0} \dim(M_\lambda^+[-n])\cdot q^n = \frac{1}{\prod_{k>0}(1-q^k)^{\dim \mathfrak{g}_{-k}}}$$

$M_\lambda^+ \overset{\text{graded v. space}}{\xleftarrow{\sim}} \underbrace{\mathcal{U}(\mathfrak{n}_-)}_{\text{PBW basis}} \overset{\text{graded v. space}}{\simeq} S(\mathfrak{n}_-)$ and it's clear that right-hand side of the above equality is the character of $S(\mathfrak{n}_-)$

$\mathfrak{n}_- = \mathfrak{g}_{-1}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-3}\oplus\ldots$

Lemma 2 / Exercise: In the particular case of $\mathfrak{g} = \mathcal{A}$ $\left(\begin{array}{l} \text{with} \\ \deg(a_n) = n \\ \deg(K) = 0 \end{array} \right)$

$$\boxed{M^+_{(1,\mu)} \xrightarrow{\sim} F_\mu = \mathbb{C}[x_1, x_2, x_3, \ldots]}$$

$\nearrow$ positive Verma ; $\uparrow$ Fock from Tuesday

$v^+_{(1,\mu)} \longmapsto 1.$

$(1,\mu)$: $\mathcal{A}_0 \longrightarrow \mathbb{C}$

$\mathcal{A}_0 = \mathbb{C}\cdot K + \mathbb{C}\cdot a_0$

$K \longmapsto 1$

$a_0 \longmapsto \mu.$

Recall: given $\mathfrak{g}$-modules M, N, a $\mathfrak{g}$-invariant pairing

$M \otimes N \longrightarrow \mathbb{C}$ s.t. $\boxed{(xm, n) + (m, xn) = 0}$ $\forall \begin{array}{l} x \in \mathfrak{g} \\ m \in M \\ n \in N \end{array}$

$(\, , \,)$

(if $M = N = \mathfrak{g} \circlearrowleft^{\text{adjoint}} \mathfrak{g}$, recover invariant $\mathfrak{g}$-form on $\mathfrak{g}$)

Alternatively, the map

$$\boxed{\begin{array}{l} m \otimes n \longmapsto (m,n) \\ M \otimes N \longrightarrow \mathbb{C} \circlearrowleft \mathfrak{g} \text{ acts trivially} \end{array}}$$

is a homomorphism of $\mathfrak{g}$-repr-s.

Prop 1: Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie alg, $\lambda \in \mathfrak{h}^*$.
There is a unique (up to a scalar) $\mathfrak{g}$-invariant pairing

$$\boxed{M_\lambda^+ \times M_{-\lambda}^- \longrightarrow \mathbb{C}}$$

Moreover, it is of degree zero, i.e.

$$\boxed{(x, y) = 0 \text{ unless } m+n=0. \quad x \in M_\lambda^+[m],\ y \in M_{-\lambda}^-[n]}$$

(x: deg m, y: deg n)

Notationwise: Let $(\cdot, \cdot)_\lambda : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ be such pairing fixed by $(v_\lambda^+, v_{-\lambda}^-) = 1$.

Lemma 3 (Hwk 2 Problem): a) Let $\mathfrak{a}$ be a Lie alg. $\mathfrak{b} \subseteq \mathfrak{a}$ – Lie subalg
$\mathfrak{b} \curvearrowright M$, $\mathfrak{a} \curvearrowright N$. Then

$$\boxed{\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M) \otimes N \underset{\mathfrak{a}\text{-mod}}{\simeq} \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(M \otimes \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{a}}(N))}$$

b) Let $\mathfrak{c}$ – Lie alg, $\mathfrak{a}, \mathfrak{b}$ – Lie subalg, $\mathfrak{c} = \mathfrak{a} + \mathfrak{b} \Rightarrow \mathfrak{a} \cap \mathfrak{b}$ Lie subal.
$\mathfrak{b} \curvearrowright M$. Then:

$$\boxed{\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{c}}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{c}}(M)) \simeq \mathrm{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}}(\mathrm{Res}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{b}}(M))}$$

Proof of Prop 1

Need to show it's 1-dim

$$\mathrm{Hom}_{\mathfrak{g}}(\underbrace{M_\lambda^+}_{\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}(\mathbb{C}_\lambda)} \otimes M_{-\lambda}^-, \mathbb{C}) \overset{\text{Lemma 3(a)}}{=} \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}(\mathbb{C}_\lambda \otimes \mathrm{Res}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}} M_{-\lambda}^-), \mathbb{C})$$

|| Frobenius reciprocity

$$\mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda \otimes \underbrace{\mathrm{Res}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}(M_{-\lambda}^-)}_{=\ \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_{-\lambda})}, \mathbb{C})$$

By Lemma 3(b): (applied to $\mathfrak{L}=\mathfrak{g}$, $\mathfrak{H}=\mathfrak{h}\oplus\mathfrak{n}_-$, $\mathfrak{K}=\mathfrak{h}\oplus\mathfrak{n}_+$, $\mathfrak{K}\cap\mathfrak{H}=\mathfrak{h}$)

$$\mathrm{Res}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}}(\mathbb{C}_{-\lambda})) \longrightarrow \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_{-\lambda})$$

||

$$\mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\underbrace{\mathbb{C}_\lambda \otimes \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_{-\lambda})}, \mathbb{C}) \overset{\text{Lemma 3(a)}}{=} \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}(\underbrace{\mathbb{C}_{+\lambda}}_{\mathrm{Res}_{\mathfrak{h}}\mathbb{C}_\lambda} \otimes \mathbb{C}_{-\lambda}), \mathbb{C})$$

|| Frob. reciprocity

$$\mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}, \mathbb{C}) \simeq \mathbb{C} \longleftarrow \text{1-dim.}$$

✓

! **Rmk 1:** Same argument shows $\nexists$ $\mathfrak{g}$-inv. pair. on $M_\lambda^+ \otimes M_\mu^-$ unless $\lambda+\mu=0$.

Rmk 2: Easy to see that above isom. sends $\mathfrak{g}$-inv. pairing to $(v_\lambda^+, v_{-\lambda}^-) \in \mathbb{C}$.

Still need to show this pairing is of degree 0.

Have: $(x m, n) = (m, -x \cdot n)$ $\forall x \in \mathfrak{g}$, $m \in M_\lambda^+$, $n \in M_{-\lambda}^-$

$\Downarrow$

$(x m, n) = (m, S(x) n)$, $S : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ – anti-automorphism given by $x \mapsto -x$. (antipode)

$((xym, n) = ~~(m, (-y)(-x)n~~)$

Assume our pairing was not of degree zero.

$x \in \mathcal{U}(\mathfrak{n}_-)$, $y \in \mathcal{U}(\mathfrak{n}_+)$

$(S(y) x v_\lambda^+, v_{-\lambda}^-) = (x v_\lambda^+, y v_{-\lambda}^-) = (v_\lambda^+, S(x) y v_{-\lambda}^-)$

(x: deg $= -n$, y: deg $= m$)

$n \neq m$

→ if $n < m \Rightarrow S(y)x$ has positive degree $\Rightarrow S(y) x v_\lambda^+ = 0$

→ if $n > m \Rightarrow S(x) y v_{-\lambda}^- = 0$ ✓

So: $(x v_\lambda^+, y v_{-\lambda}^-) = 0$ for $\deg(x) + \deg(y) \neq 0$.

Remains to use $M_\lambda^+[-n] \leftarrow \mathcal{U}(\mathfrak{n}_-)[-n]$, $M_{-\lambda}^-[m] \leftarrow \mathcal{U}(\mathfrak{n}_+)[m]$. ☐

Theorem 1: Assuming that $\mathfrak{g}$ is a non-degen. $\mathbb{Z}$-graded Lie alg.,

then for any $n>0$, the form

$$\boxed{(\,,\,)_\lambda \big|_{M_\lambda^+[n] \times M_\lambda^-[-n]} \longrightarrow \mathbb{C}}$$

is nondegenerate for generic $\lambda \in \mathfrak{h}^*$

We will prove Next week

Application: Irreducible modules

$(\, , \,)_\lambda : M_\lambda^+ \times M_{-\lambda}^- \longrightarrow \mathbb{C}$.

Let $J_\lambda^\pm \subseteq M_\lambda^\pm$ be the kernels of that form.

As $(\, , \,)_\lambda$ is of degree 0 $\Rightarrow$ $J_\lambda^\pm$ are actually $\mathbb{Z}$-graded submodules

(b/c $x = \sum x_{-n} \in M_\lambda^+ \Rightarrow (x, y_m) = (x_{-m}, y_m)$)

$y_m \in M_{-\lambda}^-[m]$

$$\Downarrow$$

$$\boxed{L_\lambda^\pm := M_\lambda^\pm / J_\lambda^\pm} \quad \leftarrow \mathbb{Z}\text{-graded } \mathfrak{g}\text{-module}.$$

$$\Downarrow$$

$$\boxed{(\cdot , \cdot)_\lambda : L_\lambda^+ \times L_{-\lambda}^- \longrightarrow \mathbb{C}} \quad \longleftarrow \text{non-degenerate pairing.}$$

Thm 2: a) $L_\lambda^\pm$ - irreducible $\mathfrak{g}$-module

b) $J_\lambda^\pm$ - the maximal graded proper submodule of $M_\lambda^\pm$

c) If there exists $L \in \mathfrak{h} (= \mathfrak{g}_0)$ s.t. $\operatorname{ad} L|_{\mathfrak{g}_n} = n \cdot \mathrm{Id}$, then

$J_\lambda^\pm$ - the maximal proper submodule of $M_\lambda^\pm$.

(i.e. no need for "graded" assumption")

Thm 1+2

⇓

Cor: If $\mathfrak{g}$ - non-deg. $\mathbb{Z}$-graded Lie alg, then

$M_\lambda^\pm$ - irreducible for generic $\lambda \in \mathfrak{h}^*$