02/02/2021

Lecture 5

Last time

→ $\mathbb{Z}$-graded Lie algebras (non-degenerate).

$$\boxed{\mathfrak{g} = \underbrace{\mathfrak{n}_+}_{>0} \oplus \underbrace{\mathfrak{h}}_{\mathfrak{g}_0} \oplus \underbrace{\mathfrak{n}_-}_{<0}}$$

$\mathbb{Z}$-deg : >0 $\quad \mathfrak{g}_0$ $\quad <0$

abelian

→ $M_\lambda^\pm$ (Verma)

→ $\exists!$ $\mathfrak{g}$-inv. pairing

$$\boxed{\begin{gathered} M_\lambda^+ \times M_{-\lambda}^- \xrightarrow{(\ ,\ )_\lambda} \mathbb{C} \\ (v_\lambda^+, v_{-\lambda}^-)_\lambda = 1 \end{gathered}}$$

$\boxed{\text{Thm 1}}$ ($\mathfrak{g}$-nondeg.): $\forall n$, the restriction $(\ ,\ )_\lambda|_{M_\lambda^+[n] \times M_{-\lambda}^-[n]}$ - nondeg for generic $\lambda \in \mathfrak{h}^*$

$\lambda \in \mathfrak{h}^*$

(λ - outside of hypersurface in $\mathfrak{h}^*$).

$J_\lambda^\pm \subseteq M_\lambda^\pm$ - kernels of $(\cdot,\cdot)_\lambda$ $\leftarrow$ graded submodules

$$\Downarrow$$

$$\boxed{L_\lambda^\pm := M_\lambda^\pm / J_\lambda^\pm}$$

Thm 2 : (a) $L_\lambda^\pm$ - irreducible

(b) $J_\lambda^\pm$ - max proper graded submodule of $M_\lambda^\pm$

(c) if $\exists L \in \mathfrak{h} = \mathfrak{g}_0$ s.t. $\mathrm{ad}\,L|_{\mathfrak{g}_n} = n \cdot \mathrm{Id}_{\mathfrak{g}_n} \Rightarrow J_\lambda^\pm$ - max proper submodule.

$$\begin{array}{ccc} M_\lambda^+ & \times & M_{-\lambda}^- \xrightarrow{\ (\ ,\ )_\lambda\ } \mathbb{C} \\ \cup & & \cup \\ J_\lambda^+ & & J_{-\lambda}^- \end{array}$$

$$\Downarrow$$

So: $\boxed{L_\lambda^+ \times L_{-\lambda}^- \xrightarrow{\ (\ ,\ )_\lambda\ } \mathbb{C} \text{ - non-degenerate!}}$

Cor : $M_\lambda^\pm$ - irred. for generic $\lambda \in \mathfrak{h}^*$ (though telling explicitly the conditions is quite non-trivial)

Proof of Thm 2

$$\mathfrak{g} = \underbrace{\mathfrak{n}_+ \oplus \mathfrak{h}} \oplus \mathfrak{n}_-$$

a) Assume L_λ^+ is not irreducible $\Rightarrow \exists\, 0 \subsetneq V \subsetneq L_\lambda^+$

$\Downarrow$

any $v \neq 0$
$= \underbrace{v_0 + v_{-1} + v_{-2} + \dots + v_{-s}}$

Choose $v \in V \setminus \{0\}$ s.t. s – min. possible integer

- If $s=0 \Rightarrow v$ is proportional to $v_\lambda^+ \Rightarrow$ generates all $L_\lambda^+ \Rightarrow V = L_\lambda^+$
- If $s>0 \Rightarrow \underbrace{\mathfrak{n}_+(v)}_{\subset V} = 0$ (by above assumption on s)

$\underbrace{(\ ,\ )}_{\text{nondeg.}} : L_\lambda^+ \times L_{-\lambda}^- \to \mathbb{C}$

Take any $x \in \mathfrak{n}_+$ and any $w \in L_{-\lambda}^-$ and compute

$$0 = (\underbrace{xv}_{0}, w)_\lambda = -\underbrace{(v, xw)} \Rightarrow v \text{ pairs trivially with all el-s } xw$$

$\Rightarrow v \in$ Kernel of $(\ ,\ )_\lambda$ in 1^{st} arg.

$\Rightarrow v=0$, since $(\ ,\ ): L_\lambda^+ \times L_{-\lambda}^- \to \mathbb{C}$ is nondeg b)

$M_\lambda^+ \supset J_\lambda^+$

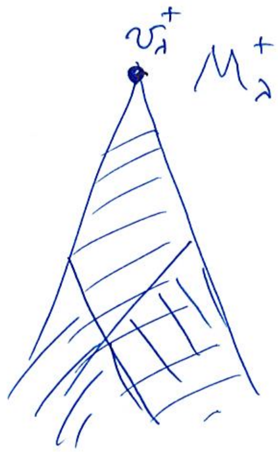
$\nexists$ (?) V - graded proper submodule

If $V \not\subseteq J_\lambda^+ \Rightarrow$ look at its image: $\overline{V} \subseteq M_\lambda^+/J_\lambda^+ = L_\lambda^+$

graded, i.e. $\overline{V} = \bigoplus \left(\overline{V} \cap L_\lambda^+[-n]\right)$.

$\overline{V} \neq 0$ - submodule of L_λ^+

By part a), $\overline{V}$ is not proper $\left(\cancel{\overline{V}=0} \text{ or } \overline{V} = L_\lambda^+ \right)$ $\Rightarrow \overline{V} = L_\lambda^+$



$$\overline{V} = L_\lambda^+$$
$$\Downarrow$$
$$\overline{v_\lambda^+} \in \overline{V}$$
$$\Downarrow !$$

Here use V-graded

$$v_\lambda^+ \in V$$
$$\Downarrow$$
$$\boxed{V = M_\lambda^+}$$

V ~~graded~~

Contradiction.

c) If $L \in \mathfrak{g}_0 = \mathfrak{h}$ s.t. $\mathrm{ad}\, L|_{\mathfrak{g}_n} = n \cdot \mathrm{Id}_{\mathfrak{g}_n}$ $\Rightarrow$ can discard "graded" in (b).

<u>Reason</u>: This L defines a grading !!!

$$M_\lambda^+ \supseteq V \circlearrowleft \mathrm{ad}(L)$$
$$\circlearrowleft \mathfrak{g} \qquad \circlearrowleft \mathfrak{g}$$

<u>Claim</u>: If $v = v_0 + v_{-1} + v_{-2} + \dots + v_{-s}$ $\in V$

$\Downarrow$

$\forall v_i \in V$

<u>Idea</u>: $\mathrm{ad}(L)^k v \in V \quad \forall\, 0 \le k \le s$

$\Downarrow$ Vandermonde determinant as all eigenvalues are different

No L as in c) ~~anywhere~~

c) holds

~~not here~~

<u>Rmk</u>: $\mathcal{A}$, $\underline{Vir}$, $\mathfrak{g}$-simple, ~~$\mathfrak{g}$-simple~~, $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\cdot K$.

$$Vir_n = \begin{cases} \mathbb{C}\cdot L_n, & n \neq 0 \\ \mathbb{C}\cdot L_0 + \mathbb{C}\cdot K, & n = 0. \end{cases}$$

<u>Vir</u>: $\mathrm{ad}(L_0)(L_n) = [L_0, L_n] = -n \cdot L_n \Rightarrow$ $\boxed{L = -L_0 \text{ plays the role of } L!}$

<u>$\mathfrak{g}$</u>: $\mathfrak{g}_1 = \mathrm{span}\{e_{\alpha_i}\}$ simple root generator $\quad \mathfrak{g}_n = \mathrm{span}\{e_\alpha \mid \alpha\text{-root of height } n\}$. $\quad \mathfrak{g}_0 = $ Cartan

$(\rho^\vee, \alpha_i) = 1$.

$\Downarrow$

$\Rightarrow$ $\boxed{\text{Take } L = \rho^\vee}$

Sketch of proof of Thm 1

Last time: $M^+_\lambda \xleftarrow{\sim} \mathcal{U}(\mathfrak{n}_-)$ $\quad$ $M^-_{-\lambda} \xleftarrow{\sim} \mathcal{U}(\mathfrak{n}_+)$

$x(v^+_\lambda) \longleftarrow x$ $\quad$ $y(v^-_{-\lambda}) \longleftarrow y$

$\mathbb{Z}$-graded v. space isom.

Thm 1: $\boxed{(\;,\;)_\lambda \Big|_{\underbrace{M^+_\lambda[-n]}_{\mathcal{U}(\mathfrak{n}_-)[-n]} \times \underbrace{M^-_{-\lambda}[n]}_{\mathcal{U}(\mathfrak{n}_+)[n]}}}$

$(xv, w) = (v, -xw)$ $\forall x \in \mathfrak{g}$

$\Downarrow$

$\boxed{(v, S(x)yw) = (xv, yw) = (S(y)xv, w) \quad x, y \in \mathcal{U}(\mathfrak{g})}$

Here: $S: \mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ – antiautom. $\quad \mathfrak{g} \ni x \mapsto -x$

can view as

$(\;,\;): \mathcal{U}(\mathfrak{n}_-)[-n] \times \mathcal{U}(\mathfrak{n}_+)[n] \longrightarrow \mathbb{C}$

$x \qquad y \longmapsto (xv^+_\lambda, yv^-_{-\lambda})_\lambda = (\underbrace{S(y)x}_{\text{of deg } 0} v^+_\lambda, v^-_{-\lambda})_\lambda$

$S(y)x\, v^+_\lambda = \pi(S(y)x)\, v^+_\lambda$

$\mathfrak{n}_+(v^+_\lambda) = 0$

$= \boxed{\lambda(\pi(S(y)x))}$

Harish Chandra projection π: $\mathcal{U}\mathfrak{g}[0] \to \mathcal{U}\mathfrak{h}$

$\boxed{\text{PBW}} \Rightarrow \mathcal{U}\mathfrak{g} \underset{\mathbb{Z}\text{-graded v. spaces}}{\cong} \mathcal{U}\mathfrak{n}_- \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}_+$

Example 1 ($\mathfrak{g} = \mathfrak{sl}_2$):

$$\mathfrak{g}_1 = e, \quad \mathfrak{g}_{-1} = f, \quad \mathfrak{g}_0 = h \rightsquigarrow \boxed{\begin{array}{l} \mathcal{U}(\mathfrak{n}_+)[n] = \mathbb{C}\cdot e^n \\ \mathcal{U}(\mathfrak{n}_-)[-n] = \mathbb{C}\cdot f^n. \end{array}} \leftarrow \text{1-dim!}$$

$$(\ ,\ )_\lambda|_{\mathbb{C}f^n \times \mathbb{C}e^n} = (f^n v_\lambda^+, e^n v_{-\lambda}^-) = \underbrace{((-e)^n f^n v_\lambda^+, v_{-\lambda}^-)}$$

$$\| \qquad \lambda = \lambda(h)$$

$$\underbrace{(-1)^n \cdot n! \cdot \lambda(\lambda-1)\ldots(\lambda-n+1)}$$

$$\underbrace{e^n f^n v_\lambda^+}$$

$$ef^n v_\lambda^+ = (fef^{n-1} + hf^{n-1})v_\lambda^+ = (f^2ef^{n-2} + fhf^{n-2} + hf^{n-1})v_\lambda^+ = \ldots$$

$$= (hf^{n-1} + fhf^{n-2} + \ldots + f^{n-2}\underbrace{hf}_{fh-2f} + \underbrace{f^{n-1}h}_{\lambda\cdot f^{n-1}v_\lambda^+})v_\lambda^+$$

$$\ldots \quad (\lambda-4)f^{n-1}v_\lambda^+ \quad (\lambda-2)f^{n-1}v_\lambda^+$$

$$= \underline{n\cdot(\lambda-(n-1))}f^{n-1}v_\lambda^+$$

$\|$ $=0$ for some n $\Leftrightarrow$ $\lambda \in \mathbb{Z}_{\geq 0}$

Upshot: $e^n f^n v_\lambda^+ = n(\lambda-n+1)\cdot e^{n-1}f^{n-1}v_\lambda^+$
$= \ldots = n!\,\lambda(\lambda-1)\ldots(\lambda-n+1)v_\lambda^+$

Upshot: M_λ^+ – irr. iff $\lambda \notin \mathbb{Z}_{\geq 0}$.

Recover the basic result about $\mathfrak{sl}_2$

$\boxed{Vir}$ Example 2 $(n=1,2)$ (general result – much later!).
for any $n \in \mathbb{Z}_{>0}$

$Vir_n = \mathbb{C} \cdot L_n$ for $n \neq 0$.

- $n = 1$: $(\ , \ )|_{\underbrace{\mathcal{U}(n_-)[-1]}_{\mathbb{C} \cdot L_{-1}} \times \underbrace{\mathcal{U}(n_+)[1]}_{\mathbb{C} \cdot L_1}}$

$$\left(L_{-1} v_\lambda^+, L_1 v_{-\lambda}^-\right) = -\big(\underbrace{L_1 L_{-1}}_{L_{-1} L_1 + 2L_0} v_\lambda^+, v_{-\lambda}^-\big) = -\left(2 L_0 v_\lambda^+, v_{-\lambda}^-\right) = -2h.$$

$\lambda \in \mathfrak{h}^* = (\mathbb{C} \cdot L_0 + \mathbb{C} \cdot \underset{\text{central}}{C})^*$ $\rightsquigarrow$ we'll incode λ by

$\lambda(L_0) =: h \leftarrow$ conformal weight
$\lambda(C) =: c \leftarrow$ central charge

So: At "depth 1" get nondegeneracy $\Leftrightarrow h \neq 0$.

• $\underline{n=2}$: $\mathcal{U}(\mathfrak{n}_-)[-2]$ has a basis $\{L_{-1}^2, L_{-2}\}$

$\mathcal{U}(\mathfrak{n}_+)[2]$ — " — $\{L_1^2, L_2\}$.

$$\det\begin{pmatrix} (L_{-1}^2 v_\lambda^+, L_1^2 v_{-\lambda}^-) & (L_{-1}^2 v_\lambda^+, L_2 v_{-\lambda}^-) \\ (L_{-2} v_\lambda^+, L_1^2 v_{-\lambda}^-) & (L_{-2} v_\lambda^+, L_2 v_{-\lambda}^-) \end{pmatrix} \neq 0.$$

Need to know when this holds!

$\Vert$ ← Exercise

$$\det\begin{pmatrix} 8h^2+4h & 6h \\ -6h & -4h-\frac{1}{2}c \end{pmatrix} = \boxed{-4h\left((2h+1)\left(4h+\frac{c}{2}\right)-9h\right)}$$

So: At "depth 2" get nondegeneracy $\Leftrightarrow \begin{cases} h \neq 0 \\ (h,c) \text{ not on parabola} \end{cases}$

! $\underline{\text{Hwk Exercise}^*}$: Finish Thm 1. (this is quite technical, so we'll skip now)

Def: For a $\mathfrak{g}$-module V, a vector $v \in V$ is called singular of weight λ if $\boxed{\mathfrak{n}_+(v) = 0, \quad x(v) = \lambda(x) \cdot v}$ for every $x \in \mathfrak{h}$

Example: $v_\lambda^+ \in M_\lambda^+$

$\boxed{\text{Lemma 1}}$: $\mathrm{Sing}_\lambda(V) \xleftarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(M_\lambda^+, V)$

$$\phi(v_\lambda^+) \longleftarrow\!\shortmid\ \phi$$

▶ $\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda, V\right) \underset{\substack{\text{Frobenius} \\ \text{reciprocity}}}{=\!=} \mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda, V) = \mathrm{Sing}_\lambda(V)$.

$\boxed{\text{Prop 1}}$ ($\mathfrak{g}$ - nondeg. / at least a), b) hold i.e. $\mathfrak{g}_n$ - f.d., $\mathfrak{g}_0$ - abelian)

M_λ^+ - irred. **iff** $\nexists$ nonzero singular vectors in negative degrees.

$\boxed{\Rightarrow:}$ If $v \in \mathrm{Sing}_\mu(M_\lambda^+)$, $v \neq 0$ $\xRightarrow{\text{Lemma}}$ nonzero homomorphism

v_μ^+

$$M_\mu^+ \longrightarrow M_\lambda^+$$

$$v_\mu^+ \longmapsto v \in M_\lambda^+[-k], \quad k>0$$

$\mathrm{Im}(M_\mu^+)$ - in negative degrees

$$\Downarrow$$

$$v_\lambda^+ \notin \mathrm{Im}(M_\mu^+)$$

$\xRightarrow{M_\lambda^+\text{-irred}} \underline{\mathrm{Im} = M_\lambda^+} \Rightarrow$ ↯ Contradiction ∎

Exercise: Any nonzero homom. $M_\mu^+ \to M_\lambda^+$ is injective (follows from PBW).

$\boxed{\Leftarrow}$ No sing. vectors $\Rightarrow$ M_λ^+ - irred.

Assume not $\Rightarrow$ $0 \subsetneq V \subsetneq M_\lambda^+$

$\Rightarrow$ $\boxed{\exists v \text{ s.t. } \boxed{\mathcal{U}(\mathfrak{g})v \subsetneq M_\lambda^+}}$

$v = v_0 + v_{-1} + \ldots + v_{-s}$

$\boxed{\text{Exercise: Can assume } v\text{ - homog.}}$ $\Rightarrow$ v is of degree $-s$

Assume that we picked v with minimal value of $s \in \mathbb{Z}_{\geq 0}$.
(i.e. v has the largest possible $\mathbb{Z}$-degree)

First: $\boxed{\mathfrak{n}_+(v) = 0}$ if not take $v' = x(v)$ for some $x \in \mathfrak{n}_+$
$\Rightarrow \mathcal{U}(\mathfrak{g})v' \subsetneq M_\lambda^+$ but v' has smaller s.

Let's look at the action of $\mathfrak{h}$.

Take all v as above with fixed s.

$\Rightarrow$ though v itself is not necessarily eigenvector for $\mathfrak{h}$

we can find v' having the same s and s.t. $x(v') = \mu \cdot v'$)

$\Rightarrow$ W Contradiction

$\boxed{\text{Cor}}$: M_λ^+ - irreducible $\iff \det(\ ,\ )|_{M_\lambda^+[-n] \times M_{-\lambda}^-[n]} \neq 0 \quad \forall n > 0$

$\iff$ no homog. singular vectors of degrees < 0.

Def: A $\mathfrak{g}$-module V is a highest-weight module of weight λ if it is generated by a critical vector of weight λ.

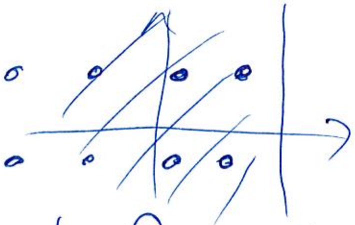
$\Updownarrow$

$$\boxed{\begin{array}{ccccc} M_\lambda^+ & \twoheadrightarrow & V & \twoheadrightarrow & L_\lambda^+ \\ v_\lambda^+ & \longrightarrow & v \in \mathrm{Sing}_\lambda(V) & & \end{array}}$$

lowest-weight: $\boxed{M_{-\lambda}^- \twoheadrightarrow V \twoheadrightarrow L_{-\lambda}^-}$

Categories $\mathcal{O}^\pm$

$$V = \bigoplus_{k\in\mathbb{C}} V[k]$$



def

Def: The objects of $\mathcal{O}^+$ are $\mathbb{C}$-graded $\mathfrak{g}$-modules s.t.

(a) all degrees lie in half-plane $\mathrm{Re}(k) < a$. (for some a) and fall into finitely many arithmetic progressions with step $= -1$.

(b) All $V[k]$ are fin. dimensional.

$\mathcal{O}^-$ – same way, but
- $\mathrm{Re}(k) > a$
- arithm. progressions with step $+1$.

M_λ^+

v_λ^+ ⟵ in any degree a

$\Downarrow$

$\uparrow$ $\mathfrak{n}_-$ $\downarrow$

examples $\mathbb{Z}_{\leq 0}$-graded

$$M_\lambda^+ = M_\lambda^+[a] + M_\lambda^+[a-1] \oplus M_\lambda^+[a-2] \oplus \dots$$

$\Downarrow$

$\downarrow$

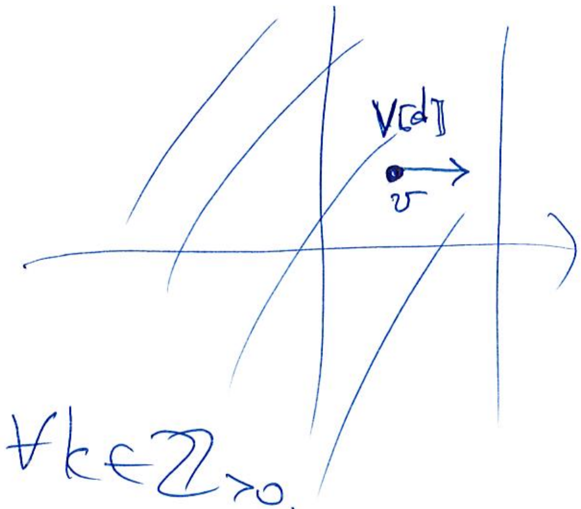
M_λ^+ satisfies conditions a) & b) with only one arithm. progression

L_λ^+

Moreover: The definition is exactly designed to fit in any finite extension of highest weight modules.

$\boxed{\text{Prop 2}}$ $L_\lambda^\pm$ - the only irreducible objects in $\mathcal{O}^\pm$ (pairwise nonisom).

Let $V \in \mathcal{O}^+$-irreducible



$$V = \bigoplus_{\mathrm{Re}\, d<a} V[d]$$

Pick d s.t. $V[d] \neq 0$ but $V[d+k] = 0$ $\forall k \in \mathbb{Z}_{>0}$.

First: $\mathfrak{n}_+(V[d]) = 0$.

Second: $\mathfrak{h} : V[d] \longrightarrow V[d] \Rightarrow$ find common eigenvector $v \in V(d)$

$\mathfrak{h} = \mathfrak{g}_0$ abelian

singular

irreducible, hence $\simeq L_\lambda^+$

Lemma $\Rightarrow$ $M_\lambda^+ \twoheadrightarrow V \simeq L_\lambda^+$

$M_\lambda^+ \ni v_\lambda^+ \longmapsto v \neq 0 \in V$

L_λ^+ : v_λ^+ in deg d_1

in deg d_2