Lecture #6

02/04/2021

Last time

$\rightarrow$ Proved Thm 1.

$$(v_\lambda^+, v_{-\lambda}^-) = 1$$

$$M_\lambda^+ \times M_{-\lambda}^- \xrightarrow{(\ ,\ )_\lambda \text{ - unique } \mathfrak{g}\text{-inv}} \mathbb{C}$$

For any n, generic λ, the restriction to "depth n" components is nondeg.

Idea: $M_\lambda^+ \xleftarrow{\sim} \mathcal{U}(\mathfrak{n}_-)$, $M_\lambda^- \xleftarrow{\sim} \mathcal{U}(\mathfrak{n}_+)$ $\leftarrow$ isom. of $\mathbb{Z}$-gr. v. spaces

$$\underbrace{\mathcal{U}(\mathfrak{n}_-)[-n]}_{\{x_i\}\text{ - basis}} \times \underbrace{\mathcal{U}(\mathfrak{n}_+)[n]}_{\{y_j\}\text{ - basis}} \longrightarrow \mathbb{C}$$

$$x,\ y \longmapsto (x v_\lambda^+, y v_{-\lambda}^-) = \underbrace{(S(y) x v_\lambda^+, v_{-\lambda}^-)}$$

$$= \lambda\Big(\underbrace{\pi(S(y)x)}_{\in}\Big)$$

$$S(\mathfrak{h}) \underset{alg.}{\simeq} \mathcal{U}(\mathfrak{h})$$

π = HC-projection of deg 0 part to $\mathcal{U}\mathfrak{h}$

Non-deg: Pick a basis ~~ext~~ $\{x_i\}$, $\{y_j\}$

$$\boxed{\det\big(\lambda(\pi(S(y_j)x_i))\big) \neq 0}$$

this condition is indep. of bases

$\longrightarrow$ Singular vectors

$\longrightarrow$ highest/lowest wt modules (i.e. $M_\lambda^\pm \twoheadrightarrow V \twoheadrightarrow L_\lambda^\pm$)

$\longrightarrow$ $\mathcal{O}^\pm$-categories (morphisms: $\mathfrak{g}$-homomorphisms preserving $\mathbb{C}$-grading!)

Def 1: For $M \in \mathcal{O}^+$ define **character** of M:

$$\boxed{ch_M(q,x) = \sum_{d\in\mathbb{C}} q^{-d}\, Tr_{M[d]}(e^x)}$$

$x \in \mathfrak{h} = \mathfrak{g}_0$.
q – formal variable

Exercise: For M_λ^+ with $v_\lambda^+ \in M_\lambda^+$ being in degree 0, we have

$$\boxed{ch_{M_\lambda^+}(q,x) = \frac{e^{\lambda(x)}}{\prod_{k>0} \det_{\mathfrak{g}_{-k}}\left(1-q^k e^{ad(x)}\right)}}$$

Hwk 3

$$\mathfrak{g}_0 \overset{ad}{\curvearrowright} \mathfrak{g}_{-k}$$

Hint: 1) A – linear operator $V \xrightarrow{} V$ (V f.dim.) $\Rightarrow$ $\boxed{\sum_{n\geq 0} q^n Tr_{S^n V}(S^n A) = \frac{1}{\det(1-qA)}}$

2) PBW ($M_\lambda^+ \simeq \mathcal{U}(\mathfrak{n}_-) \simeq S(\mathfrak{n}_-)$ as $\mathbb{Z}$-gr. v. spaces)

Def: If $V = \bigoplus_{d\in\mathbb{C}} V[d]$ $\mathbb{C}$-graded v.space & $\dim V[d] \overset{\forall d}{<} \infty$

$\rightsquigarrow$

<u>restricted dual</u> $V^\vee := \bigoplus_{d\in\mathbb{C}} V[d]^* \subset V^*$ ← entire dual (i.e. all linear functions $V \to \mathbb{C}$)

Note: $\boxed{V^{\vee\vee} \simeq V.}$ for either of the two gradings

Note: Two gradings: $\underbrace{V^\vee[d] = V[\pm d]^*.}$

Lemma 1: If $V = \bigoplus_{d\in\mathbb{Z}} V[d]$ $\mathbb{C}$-graded module over $\mathbb{Z}$-graded Lie alg. $\mathfrak{g}$

then $V^\vee$ is also a $\mathfrak{g}$-module

$\mathbb{C}$-graded via $V^\vee[d] = V[-d]^*$.

$\left(\begin{array}{c} \varphi\in V^\vee,\ x\in\mathfrak{g} \\ (x\varphi)(w) \overset{\text{def}}{=} \varphi(-x(w))\ \forall w\in V \end{array} \right)$

! <u>Check</u>: "$-$": works
"$+$": wouldn't work

► Straightforward – hwk check ■

Lemma 2: We have two mutually inverse anti-equivalences

$$\boxed{\mathcal{O}^+ \xrightarrow{\ \vee\ } \mathcal{O}^- \quad \& \quad \mathcal{O}^- \xrightarrow{\ \vee\ } \mathcal{O}^+}$$

(Here: $\phi : V \to W \rightsquigarrow \phi^\vee : W^\vee \to V^\vee$)

(Recall: $M_\lambda^+ \in \mathcal{O}^+$, $M_{-\lambda}^- \in \mathcal{O}^-$)

$$(\ ,\ )_\lambda : M_\lambda^+ \times M_{-\lambda}^- \longrightarrow \mathbb{C}$$

$$\Updownarrow$$

linear map $M_\lambda^+ \longrightarrow (M_{-\lambda}^-)^\vee$ $\longleftarrow$ $\mathbb{Z}$-graded (use that $(\cdot,\cdot)_\lambda$ was of degree 0)

$\longleftarrow$ $\mathfrak{g}$-homom. $\left(\varphi_{xv} \overset{?}{=} x\varphi_v \Leftrightarrow \underbrace{(xv, w) = (v, -xw)}_{\mathfrak{g}\text{-invariance}} \right)$

$$v \longmapsto \varphi_v : \varphi_v(w) = (v, w)_\lambda.$$

Easy: φ_v is in restricted dual

$\boxed{\text{Upshot}}$: $(\cdot,\cdot)_\lambda$ yields:

$$\boxed{M_\lambda^+ \twoheadrightarrow L_\lambda^+ \xrightarrow{\ \sim\ } (L_{-\lambda}^-)^\vee \hookrightarrow (M_{-\lambda}^-)^\vee}$$

~~∀ Any highest weight module V_λ, we also have~~

Involutions

We'll often have $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ being endowed with involutive automorph. $\omega : \mathfrak{g} \circlearrowleft$, s.t.

i.e. $\omega^2 = Id_{\mathfrak{g}}$

$$\boxed{\omega(\mathfrak{g}_k) = \mathfrak{g}_{-k}, \quad \omega|_{\mathfrak{g}_0} = -Id_{\mathfrak{g}_0}}$$

Example : 1) $\mathfrak{g} = \mathcal{A}$, $\omega : a_k \mapsto -a_{-k}$, $K \mapsto -K$

Check $\begin{cases} [a_m, a_n] = n\delta_{m,-n} \cdot K \\ \downarrow \qquad\qquad \downarrow \\ [-a_{-m}, -a_{-n}] \qquad n\delta_{m,-n}(-K) \\ \shortparallel \qquad\qquad \shortparallel \\ [a_{-m}, a_{-n}] = -n\delta_{m,-n}K \end{cases}$

2) $\mathfrak{g} = Vir$, $\omega : L_k \mapsto -L_{-k}$, $C \mapsto -C$

3) $\mathfrak{g}$ - simple, f.d.; $\omega : \mathfrak{g} \to \mathfrak{g}$ is given $e_i \mapsto f_i$, $f_i \mapsto e_i$, $h_i \mapsto -h_i$.

4) KM $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} \cdot C$; $\omega : at^k \mapsto \overset{(3)}{\omega}(a) t^{-k}$, $K \mapsto K$

! Check at home that ω is indeed an automorphism in all examples above.

$\mathfrak{g} \curvearrowright M = \bigoplus_{d \in \mathbb{C}} M[d]$

$\circlearrowleft_{\omega}$ $\rightsquigarrow$ twist M by ω.

provides equivalences of categories

$$\boxed{\mathcal{O}^+ \xrightarrow{\ \omega\ } \mathcal{O}^- \quad \& \quad \mathcal{O}^- \xrightarrow{\ \omega\ } \mathcal{O}^+}$$

Note: to make this work need to take opposite grading! (since ω maps $\mathfrak{g}_k$ to $\mathfrak{g}_{-k}$)

Combining $^\vee$ & ω, we get:

$$\begin{tikzcd} \mathcal{O}^+ \arrow[r, "\vee"] \arrow[rr, bend right, "c"'] & \mathcal{O}^- \arrow[r, "\omega"] & \mathcal{O}^+ \end{tikzcd}$$

!

$$\boxed{\begin{array}{ccc} M & \longmapsto & M^c \\ \cap & & \cap \\ \mathcal{O}^+ & & \mathcal{O}^+ \end{array}}$$

$\leftarrow$ "contragredient module"

<u>Obvious</u> (Easy Exercise):

$$\boxed{\begin{array}{ccc} M_\lambda^+ & \xrightarrow{\sim} & (M_{-\lambda}^-)^\omega \quad \text{as } \mathfrak{g}\text{-modules} \\ \boxed{v_\lambda^+} & \longmapsto & v_{-\lambda}^- \\ \underset{U(\mathfrak{h}\oplus\mathfrak{n}_+)}{x} \otimes v_\lambda^+ & \longmapsto & \omega(x) \otimes v_{-\lambda}^- \end{array}}$$

Recall $\underline{\mathfrak{g}\text{-inv}}$ $(\cdot,\cdot)_\lambda : M_\lambda^+ \times M_{-\lambda}^- \longrightarrow \mathbb{C}$

$(xv, w) + (v, xw) = 0$

$$\Big\Updownarrow$$

$$M_\lambda^+ \times (M_\lambda^+)^\omega \longrightarrow \mathbb{C}.$$

$$\Big\Updownarrow$$

$(v_\lambda^+, v_\lambda^+) = 1.$ (1)

So: Can decode $(\cdot,\cdot)_\lambda$ as follows:

$$(\cdot,\cdot) : M_\lambda^+ \times M_\lambda^+ \longrightarrow \mathbb{C}$$

Benefit: Single Verma in game!

contravariant : $\boxed{(xv, w) + (v, \omega(x)w) = 0.}$ (2)

Later in the course : Criterias for irr. Verma / KM, Vir

$\Uparrow$

"determinant f-las"

(those are exactly in terms of contravariant form)

Lemma 3 : The form $(\cdot,\cdot)$ is symmetric!

▷ The transpose form (swapping two copies of M_λ^+) also satisfies (1) & (2) $\Rightarrow$ coincides with $(\ ,\ )$ by uniqueness. ∎

Conclusion: $\exists!$ contravariant form $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$
which symmetric $(v_\lambda^+, v_\lambda^+) = 1$

factors through kernels I_λ^+

So: $\forall$ h.wt λ module V also have $\exists!$ contrav. form

$$\boxed{V \times V \xrightarrow{(\cdot,\cdot)} \mathbb{C} \qquad (v,v) = 1}$$

h.wt. vector

! Let me state once again the contravariance condition:

$(xv, u) + (v, \omega(x)u) = 0 \quad \forall u,v \in V \ \forall x \in \mathfrak{g}$, where $\omega: \mathfrak{g} \to \mathfrak{g}$ - our involutive action.

For what follows, need to recall

unitary repr-s (we'll do it briefly).

Setup

- $\mathfrak{g}$ - Lie alg/$\mathbb{C}$
- $\dagger : \mathfrak{g} \to \mathfrak{g}$ - antilinear anti-involution $\left(\begin{array}{l} \text{i.e: } \dagger^2 = \mathrm{Id},\ [a,b]^\dagger = -[a^\dagger, b^\dagger] \\ \forall \lambda \in \mathbb{C},\ a \in \mathfrak{g} : (\lambda a)^\dagger = \bar{\lambda} a^\dagger \end{array} \right)$

$$\boxed{\mathfrak{g}_{\mathbb{R}} := \{ a \in \mathfrak{g} \mid a^\dagger = -a \}}$$

v. subspace /$\mathbb{R}$ which is a Lie alg /$\mathbb{R}$

$\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$

$\boxed{\text{Hwk 3}}$

Def 3: Let $\mathfrak{g}$ be a Lie alg$/\mathbb{C}$ with real structure $\dagger$, and let V be a $\mathfrak{g}$-module. Then:

* V – Hermitian if it's equipped with nondeg. Hermitian form s.t. $(av, w) = (v, a^\dagger w)$ $\forall a \in \mathfrak{g}$ $\forall v, w \in V$.
* V – unitary if $(\cdot,\cdot)$ is positive definite.

Rmk: $\mathfrak{g}_{\mathbb{R}}$ acts by skew-hermitian matrices.

In our setup we'll always assume that: $\boxed{\mathfrak{g}_k^\dagger = \mathfrak{g}_{-k}}$ $\forall k$.

$\dagger \curvearrowright \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$

in particular

$\boxed{\dagger : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0}$

$\Downarrow$

can look at $\boxed{\mathfrak{g}_{0\mathbb{R}} = \{a \in \mathfrak{g}_0 \mid a^\dagger = -a\}}$

Lemma 4/Exercise: In the above setup, let $\lambda \in \mathfrak{h}^*$ be real ($\lambda \in \mathfrak{g}^*_{0\mathbb{R}}$, equiv: $\overline{\lambda(a^+)} = -\lambda(a)$ $\forall a \in \mathfrak{g}_{00}$)

Then $M_\lambda^+ = M_\lambda$ has a unique Hermitian form $(\cdot,\cdot)$ with $(v_\lambda^+, v_\lambda^+) = 1$.

Rmk: If λ is not real $\Rightarrow$ get an issue with

$x \in \mathfrak{h}$

$\boxed{\text{Examples}}$:

1) $\mathfrak{g} = \mathcal{A}$: $a_k^+ = a_{-k}$, $K^+ = K$

2) $\mathfrak{g} = Vir$: $L_k^+ = L_{-k}$, $C^+ = C$

3) $\mathfrak{g}$ - simple $e_i^+ = f_i$, $f_i^+ = e_i$, $h_i^+ = h_i$

4) $\hat{\mathfrak{g}}$ $(at^k)^+ = a^+ t^{-k}$, $K^+ = K$.

! Exercise - verify all those examples

Semidirect product of Lie alg-s

We shall now start a new topic, following [Kac-Raina] in the next few weeks!

Setup
- $\mathfrak{g}, \mathfrak{a}$ — Lie alg-s
- $\mathfrak{g} \xrightarrow{\rho} \underbrace{\mathrm{Der}(\mathfrak{a})}_{\text{derivations}}$ — Lie alg. homom.

$\Downarrow$

Cook out $\boxed{\mathfrak{g} \ltimes \mathfrak{a}}$ (semidirect product) — Lie alg, which is defined as follows:

- as a vector space it's $\mathfrak{g} \oplus \mathfrak{a}$
- bracket: $\boxed{[(\overset{\mathfrak{g}}{x}, \overset{\mathfrak{a}}{\alpha}), (\overset{\mathfrak{g}}{y}, \overset{\mathfrak{a}}{\beta})] = ([x,y], [\alpha,\beta] + \overset{\rho(x)(\beta)}{x(\beta)} - y(\alpha))}$

Rmk: $\mathfrak{g} \hookrightarrow \mathfrak{g} \ltimes \mathfrak{a} \hookleftarrow \mathfrak{a}$ — Lie alg. homom.

$$\mathfrak{g} \ltimes \mathfrak{a} \twoheadrightarrow \mathfrak{g}$$

Rmk: 1) El-s of $\mathfrak{g}$ commuting as before

2) El-s of $\mathfrak{a}$ commuting as before

3) $\mathfrak{g}$ commutes with $\mathfrak{a}$ via

$$[\underbrace{(x,0)}_{x}, \underbrace{(0,\beta)}_{\beta}] = [0, x(\beta)]$$

$$\boxed{[x, \beta] = x(\beta).}$$

Lecture 1: $W \longrightarrow Der(\mathcal{A})$

$$f\partial_t \longmapsto f\partial_t(g,\alpha) = (fg', 0)$$

Explicitly in terms of basis: $\boxed{L_n(a_m) = -m\, a_{n+m}.}$

(here, $L_n = -t^{n+1}\partial_t$)

$\leadsto \underset{\text{Witt}}{W} \ltimes \underset{\text{Heisenberg}}{\mathcal{A}}$

Also: $\mathcal{A} \curvearrowright F_\mu$ – Fock module.

$\boxed{Q}$: Does it extend to $W \ltimes \mathcal{A} \curvearrowright F_\mu$?

$$\Updownarrow$$

(a) Are there operators $L_n \in End(F_\mu)$ s.t. $[L_n, a_m] = -m a_{n+m}$

(b) Do they also satisfy $[L_n, L_m] = (n-m) L_{n+m}$?

$\boxed{A}$: Yes for (a)

No for (b) $\longleftarrow$ Corrected by $Vir \ltimes \mathcal{A}$ $\qquad Vir \xrightarrow{C\mapsto 0} W \curvearrowright \mathcal{A}$

Lemma 4 : For every $n \in \mathbb{Z}$, there is a unique (up to scalar op-s) operator $L_n : F_\mu \circlearrowleft$ s.t. $\boxed{[L_n, a_m] = -m a_{n+m} \quad \forall m.}$

1) Uniqueness is easy: Assume $L_n^{(1)}, L_n^{(2)}$ satisfy

$\Downarrow$

$L_n^{(1)} - L_n^{(2)}$ commutes $\underbrace{\text{with all } a_m}_{\text{with } A}$

But: F_μ - irred. A-module $\xRightarrow{\text{Dixmier's Lemma}} L_n^{(1)} - L_n^{(2)} = \lambda \cdot Id_{F_\mu}$. ✓

2) Construction of L_n

Def: For $m, n \in \mathbb{Z}$, the normally ordered product $:a_m a_n: \in U(A)$:

$$\boxed{:a_m a_n: = \begin{cases} a_m a_n & \text{if } m \le n \\ a_n a_m & \text{if } m > n. \end{cases}}$$

Rmk: 1) $m+n \neq 0 \Rightarrow :a_m a_n: = a_m a_n$

$\Downarrow$

2) $[:a_m a_n:, x] = [a_m a_n, x]$

$n > 0$

$a_n a_{-n} = :a_n a_{-n}: + n \cdot K.$

Define

$$\boxed{L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} :a_j a_{j+n}: \quad \forall n\in\mathbb{Z}} \in \mathcal{U}(\mathcal{A})^{\wedge}$$

completion

F_μ is well-defined. (since $\forall v\in F_\mu \ \exists N$ s.t. $a_{>N}(v)=0$)

Explicitly: ~~For $j\neq 0$~~ :

$$\boxed{\begin{aligned} &L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_j a_{j+n} \quad \text{for } n\neq 0 \\ &L_0 = \frac{\mu^2}{2} + \underbrace{\sum_{k>0} a_{-k}a_k}_{\text{Euler field}} \end{aligned}}$$

~~For $j=0$~~ :

Suffices to show: $[L_n, a_m] = -m a_{n+m}$.

prove on the next page

$$[L_n, a_m] = \Big[\frac{1}{2}\sum_j :a_{-j}\,a_{j+n}:,\ a_m\Big] = \frac{1}{2}\sum \underbrace{[a_{-j}\,a_{j+n},\ a_m]}_{[a_{-j},a_m]\,a_{j+n} + a_{-j}\,[a_{j+n},a_m]}$$

$$= \underbrace{\frac{1}{2}\sum_j \underbrace{[a_{-j}, a_m]}_{=0 \text{ unless } j=m} a_{j+n}}_{-\frac{1}{2} m\cdot K\cdot a_{m+n}} + \underbrace{\frac{1}{2}\sum_j a_{-j}\underbrace{[a_{j+n}, a_m]}_{=0 \text{ unless } j=-m-n}}_{-\frac{1}{2} m\, a_{m+n} K}.\ =$$

$$K \overset{\mathrm{id}}{\curvearrowright} F_\mu \qquad \overset{\text{on } F_\mu}{=} \boxed{-m\, a_{m+n}}.$$

$\underline{\underline{\text{So}}}$: Positive answer to (a)!

Lemma 5: $[L_n, L_m] = (n-m) L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m} \mathrm{Id}_{F_\mu}$.

$\Downarrow$ $\Uparrow$

$\boxed{\text{Cor: } \quad Vir \ltimes A \curvearrowright F_\mu}$

Direct proof $\leftarrow$ Hwk (try to do this directly!)

$\leftarrow$ [Kac-Raina, Prop 2.3]

Let's present a bit less technical proof

- Step 1: $[L_n, L_m] - (n-m) L_{n+m}$ — scalar on $F_\mu \circlearrowleft A$.

$$[[L_n, L_m] - (n-m) L_{n+m}, a_k] = \underbrace{[\underbrace{[L_n, a_k]}_{-k a_{k+n}}, L_m]}_{\substack{-k(k+n) a_{k+m+n} \\ -k^2 - kn}} + \underbrace{[L_n, \underbrace{[L_m, a_k]}_{-k a_{k+m}}]}_{k(k+m) a_{k+m+n}} \underbrace{- (n-m) [L_{n+m}, a_k]}_{+(n-m)k \cdot a_{k+m+n}}$$

$$= 0 \cdot a_{k+m+n}.$$

Dixmier's lemma

Step 2

$n+m \neq 0 \Rightarrow$ constant must be 0!

Indeed: L_k - degree k-operator $\Rightarrow [L_n, L_m] - (n-m)L_{n+m}$ - of deg $n+m$ - scalar

$\Rightarrow$ ZERO if $n+m \neq 0$.

Step 3

Wait: $\gamma_n = \frac{n^3-n}{12}$

Rename: Find γ_n, const s.t. $[L_n, L_{-n}] - 2n L_0 = \gamma_n \cdot Id$

Consider the map $W \times W \longrightarrow \mathbb{C}$, $(L_n, L_m) \longmapsto \gamma_n \cdot \delta_{n,-m}$ $\leftarrow$ must be a 2-cocycle.

As $H^2(W) = 1 \Rightarrow \exists c \in \mathbb{C}$, $\exists \xi \in W^*$ s.t. $\gamma_n \delta_{n,-m} = c \cdot \frac{n^3-n}{12} + \xi([L_n, L_m])$

$$\left.\begin{aligned} L_0(1) &= \frac{\mu^2}{2} \\ L_{\cdot}(1) &= \mu^2 \end{aligned}\right\} \Rightarrow \xi = 0$$

$$\boxed{\begin{aligned} &\Rightarrow \xi = 0 \\ &\Rightarrow c = 1 \end{aligned}}$$

$\vdots$

$L_2(1) = 0 \Rightarrow [L_2, L_{-2}](1) = 2\mu^2 + \frac{1}{2} \Rightarrow c = 1.$

... (More details next time!)