02/09/2021

Lecture 7

Last time:

1) Restricted dual: $\mathcal{O}^+ \overset{\checkmark}{\hookrightarrow} \mathcal{O}^-$, $\mathcal{O}^- \hookrightarrow \mathcal{O}^+$

2) $\omega : \mathfrak{g} \circlearrowleft$ involutive action, s.t. $\mathfrak{g}_k \mapsto \mathfrak{g}_{-k}$, $\omega|_{\mathfrak{g}_0} = -\mathrm{Id}_{\mathfrak{g}_0}$

$\rightsquigarrow$ Replace $\mathfrak{g}$-inv. $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$

$\Downarrow$

$\boxed{\text{contravariant form } M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}}$

3) Hermitian form (Input: $\mathfrak{g}$ – $\mathbb{Z}$-graded $\mathbb{C}$-Lie alg; $\dagger : \mathfrak{g} \circlearrowleft$ – antilinear anti-involution)

to be recalled later today...

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4) $\mathfrak{g}, \mathfrak{a}$ – Lie alg-s, $\mathfrak{g} \xrightarrow[\text{hom.}]{\text{Lie alg.}} \mathrm{Der}(\mathfrak{a})$ $\rightsquigarrow$ $\boxed{\mathfrak{g} \ltimes \mathfrak{a}}$ semidirect product

as vector space it's $\mathfrak{g} \oplus \mathfrak{a}$

Lie Bracket:

$$\boxed{[(\underset{\mathfrak{g}}{X}, \underset{\mathfrak{a}}{\alpha}), (y, \beta)] = ([X, y], [\alpha, \beta] + X(\beta) - y(\alpha))}$$

Applied to

- $\mathrm{Vir} \xrightarrow{K \to 0} \underset{\text{Witt alg}}{W} \longrightarrow \mathrm{Der}(\underset{\text{Heisenberg alg}}{A})$ $\rightsquigarrow$ $\boxed{\mathrm{Vir} \ltimes A, \quad W \ltimes A}$

Also: $A \curvearrowright F_\mu$ ← Fock rep-s (as v. space $\mathbb{C}[x_1, x_2, \dots]$)

$\boxed{Q}$: Does it extend to one of the actions $\mathrm{Vir} \ltimes A$ or $W \ltimes A$ on F_μ

<u>Last time</u>:

$$\boxed{L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} :a_{-j}\, a_{j+n}:}$$

$\underbrace{:a_m a_n:}_{\text{normally ordered product}} = \begin{cases} a_m a_n, & m \le n \\ a_n a_m, & m > n. \end{cases}$

(matters only for $m+n=0$!)

<u>Last time proved</u>: 1) $\boxed{[L_n, a_m] = -m\, a_{n+m}}$ $\forall n,m$

have not verified this constant last time (see argument below)

2) $$\boxed{\underbrace{[L_n, L_m] - (n-m) L_{n+m}}_{\text{commutes with all } \{a_k\} \Rightarrow \text{ it acts by scalar on } F_\mu} = \frac{n^3-n}{12}\,\delta_{n,-m}\cdot \mathrm{Id}_{F_\mu}}$$

(scalar $=0$ if $n+m \neq 0$)

<u>Still Remains</u>: When $m=-n$, this constant γ_n is exactly $\dfrac{n^3-n}{12}$

Let's provide an argument showing this constant is $\frac{n^3-n}{12}$ without involving many computations (see Kac – Raina for a different proof).

Consider bilinear map

$W \times W \longrightarrow \mathbb{C}$

with $(L_n, L_m) \longmapsto \gamma_n \delta_{n,-m}$

Note: By the very construction it <u>must</u> be a 2-cocycle.

$H^2(W)$ - 1-dim $\rightsquigarrow$ Lecture 1

$\exists c \in \mathbb{C}, \ \exists \xi \in W^*$ s.t.

$$\boxed{\gamma_n \delta_{n,-m} = \underbrace{c \cdot \frac{n^3-n}{12}\delta_{n,-m}}_{\text{multiple of Virasoro cocycle.}} + \overbrace{\xi([L_n, L_m])}^{\text{2-coboundary}}}$$

Want: $c=1$, $\xi = 0$

- $n+m \neq 0 \Rightarrow \boxed{\xi(L_k) = 0 \text{ for } k \neq 0.}$
- $\overset{F_\mu}{\Downarrow} L_0(1) = \Big(\frac{\mu^2}{2} + \underbrace{\sum_{j>0} a_{-j} a_j (1)}_{0}\Big) = \frac{\mu^2}{2}$

$[L_1, L_{-1}](1) = (L_1 L_{-1} - L_{-1} L_1)(1) = L_1 \underbrace{L_{-1}(1)}_{\mu x_1} = L_1(\mu x_1) = \mu^2$

Plug this for $n = -m = 1$: ~~$L_{-1}(1) = \mu$~~

$\underbrace{[L_1, L_{-1}] - 2L_0}_{2\gamma_1} : 1 \longmapsto \mu^2 - 2 \cdot \frac{\mu^2}{2} = 0 \Rightarrow \boxed{\gamma_1 = 0} \Rightarrow \xi(L_0) = 0 \Rightarrow \xi = 0$

- $[L_2, L_{-2}](1) = \frac{1}{2} + 2\mu^2 \Rightarrow \underbrace{[L_2, L_{-2}] - 4L_0}_{\gamma_2 = c \cdot \frac{1}{2} + 0} : 1 \mapsto \frac{1}{2} + \cancel{2\mu^2} - \cancel{4 \cdot \frac{\mu^2}{2}} \Rightarrow \boxed{c = 1}$

Upshot : $\boxed{\mathrm{Vir} \ltimes \mathcal{A} \curvearrowright F_\mu}$ $\rightsquigarrow$ $\boxed{\mathrm{Vir} \curvearrowright F_\mu}$

This action of Vir admits a 1-parameter family of deformation $\boxed{\mathrm{Vir} \curvearrowright F_\mu.}$ ← BUT: it's no longer a repr-n of $\mathrm{Vir} \ltimes \mathcal{A}$!

Prop 1 : Let $\lambda, \mu \in \mathbb{C}$. Define $\tilde{L}_n : F_\mu \to F_\mu$ via

Hwk 4

can be disregarded f

$$\tilde{L}_n := \frac{1}{2} \sum_{j \in \mathbb{Z}} :a_j a_{j+n}: + i \cdot \lambda \cdot n a_n , \quad n \neq 0$$

$$\tilde{L}_0 := \sum_{j > 0} a_{-j} a_j + \frac{\mu^2 + \lambda^2}{2}$$

Note: The reason why we write $i \cdot \lambda$ instead of λ will be clear when we discuss Hermitian form below.

Then :

$$[\tilde{L}_n, a_m] = -m a_{n+m} + i \lambda m^2 \delta_{m,-n}$$

$$[\tilde{L}_n, \tilde{L}_m] = (n-m) \tilde{L}_{n+m} + \delta_{n,-m} \cdot \frac{n^3 - n}{12} (1 + 12 \lambda^2)$$

! For $\lambda = 0$ this recovers the aforementioned $\mathrm{Vir} \curvearrowright F_\mu$ from last time

Conclusion: We get $Vir \curvearrowright F_\mu$ with $\boxed{K \mapsto 1+12\lambda^2}$

Next: Hermitian Forms in the above context.

Recall

$$\boxed{\begin{array}{l} \dagger : \mathcal{A} \to \mathcal{A} \quad \text{via} \quad a_k^\dagger = a_{-k} \quad K^\dagger = K \\ \dagger : Vir \to Vir \quad \text{via} \quad L_k^\dagger = L_{-k}, \quad C^\dagger = C. \end{array}}$$

$$\boxed{(av, w) = (v, a^\dagger w)}$$ ← the compatibility of the Hermitian form with $\mathfrak{g}$-action

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In our setup: $\boxed{\mathcal{A} \curvearrowright F_\mu \text{ - Fock.}}$

Recall: as an $\mathcal{A}$-module $F_\mu \simeq M_{1,\mu}$ - Verma for $\mathcal{A}$

Lemma 1: If $\mu \in \mathbb{R}$, there is a unique Hermitian form $F_\mu \times F_\mu \xrightarrow{(\cdot,\cdot)} \mathbb{C}$ w.r.t. $\mathcal{A}$-action s.t. $(1,1) = 1$.

(Follows from Thursday as Fock is Verma for $\mathcal{A}$-action).

Explicitly, we have

$$\boxed{\left(X_1^{n_1} X_2^{n_2} \cdots X_r^{n_r},\ X_1^{m_1} X_2^{m_2} \cdots X_r^{m_r}\right) = \prod_j \delta_{n_j, m_j} \cdot \prod_j m_j! \cdot \prod_j j^{m_j}}$$

As $(av, w) = (v, a^\dagger w)$, we have

$$\underbrace{X_1^{n_1} \cdots X_r^{n_r}}_{\parallel} = a_{-1}^{n_1} a_{-2}^{n_2} \cdots a_{-r}^{n_r} (1) \Rightarrow (X_1^{n_1} \cdots X_r^{n_r}, w) = (1, \underbrace{a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}}_{a_j \mapsto j\, \partial/\partial x_j} (w))$$

$w = x_1^{m_1} \cdots x_r^{m_r}$

<u>In particular</u> : $\boxed{\text{Our form is actually unitary! (i.e. positive definite).}}$

<u>Recall</u> : $\mathcal{A} \curvearrowright F_\mu \xleftarrow[\tilde{L}_n]{\text{Prop 1}} Vir$

$K = 1 + 12\lambda^2$

$\boxed{\text{Corollary 1}}$: If $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}$, then $\mathrm{Vir} \overset{\tilde{L}_n}{\curvearrowright} F_\mu$ is unitary.
(w.r.t. same form).

▶ Need to check: $(\tilde{L}_n v, w) \overset{?}{=} (v, \underbrace{\tilde{L}_{-n}}_{\tilde{L}_n^{+}} w)$

$n \neq 0$: $\tilde{L}_n^{\overset{\text{adjoint}}{*}} = \frac{1}{2} \sum_j \underbrace{(a_{-j}\, a_{n+j})^*}_{\underbrace{a_{n+j}^* \, a_{-j}^*}_{a_{-n-j}\, a_j}} + \underbrace{(in\lambda a_n)^*}_{-in\lambda \underbrace{a_n^*}_{a_{-n}}} = \tilde{L}_{-n}$

$n=0$: same computation ← Work out at home!

$\overline{\dagger : L_k \mapsto L_{-k}}$

Prop 2: Let V be a highest weight <u>unitary</u> repr-n.
Then V is irreducible.

▷

$$M_\lambda^+ \twoheadrightarrow \underbrace{V \twoheadrightarrow L_\lambda^+}_{\text{Need: it's isom.}} \qquad v \longmapsto v_\lambda^+$$

Take the kernel $\underbrace{K := \mathrm{Ker}(V \twoheadrightarrow L_\lambda^+)}$. $\rightsquigarrow$ look at $K^\perp$ w.r.t. our unitary form

<u>Then</u>:

1) $K \oplus K^\perp = V$
2) $K, K^\perp$ – submodules

$$K^\perp \hookrightarrow V \longrightarrow L_\lambda^+, \qquad K^\perp \xrightarrow{\sim} L_\lambda^+$$

V – h. wt. module.
v – h. wt. vector $\Rightarrow v \in K^\perp$

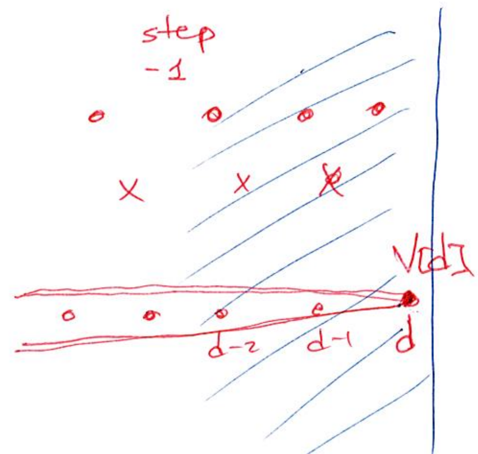
$\Rightarrow v$ does not generate all V!

$$\Rightarrow \lightning \Rightarrow K = 0 \Rightarrow \boxed{V \simeq L_\lambda^+}$$

■

Corollary 2: Any ~~repr-n~~ unitary repr-n in category $\mathcal{O}^+$ of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ is completely reducible (i.e. $\simeq \bigoplus L_{\lambda_j}$).

▶ $V \in \mathcal{O}^+$, so $V = \bigoplus_{d \in \mathbb{C}} V[d]$



Pick $d \in \mathbb{C}$ s.t. $V[d] \neq 0$

$V[d+1] = V[d+2] = \ldots = 0$.

Pick nonzero $v \in V[d] \rightsquigarrow$ generate submodule $W_1 = \mathcal{U}(\mathfrak{g})(v)$ (h. wt. module)

V-unitary $\Rightarrow$ W_1-unitary $\overset{\text{Prop 2}}{\Rightarrow}$ $W_1 \simeq L_\lambda$ & $V = W_1 \oplus W_1^{\perp}$

$\downarrow$ apply the same reasoning to $W_1^{\perp}$ etc.

$\Rightarrow$ $\boxed{W \cong \bigoplus L_{\lambda_j}}$

$F_\mu \ni 1$ - singular vector w.r.t. A but also w.r.t. Vir. !

Vir: K - central $K(1) = 1 + 12\lambda^2 \leftarrow \underline{c}$

L_0, K - basis of Vir_0 $L_0(1) = \frac{\lambda^2+\mu^2}{2} \leftarrow \underline{h}$.

Verma module for Vir

So:

$$\begin{array}{ccc} M^+_{h,c} & \xrightarrow[\pi]{\text{Vir-module homom.}} & F_\mu \\ \cup & & \cup \\ v^+_{h,c} & \longrightarrow & 1 \end{array}$$

$\deg L_k = k$

$\deg a_k = k$

Warning: Not to confuse with $F_\mu \simeq M_{1,\mu}$ as A-modules, where $M_{1,\mu}$ is Verma for A!

Lemma 2: For generic λ, μ, this homom. $M^+_{h,c} \to F_\mu$ is an isom.

- Both Verma module $M^+_{h,c}$ & Fock module F_μ are $\mathbb{Z}$-graded with fin. dim. deg n components of the same dimension equal to $p(n)$ = number of partitions.., and the map $M^+_{h,c} \to F_\mu$ preserves gradings.

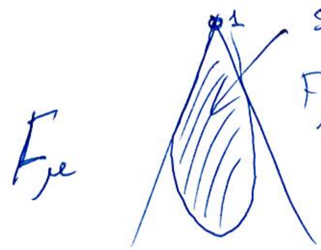
So: π - isom $\Leftrightarrow$ π - injective

Know: For generic h, c, Verma $M^+_{h,c}$ - irreducible (general result for any non-degener. $\mathbb{Z}$-graded Lie alg)

$\Rightarrow$ map is generically injective $\Rightarrow$ isom for generic λ, μ.

Always have: $M^{+}_{h,c} \xrightarrow{\pi} F_{\mu} \curvearrowleft^{\hat{L}_n} \mathrm{Vir}$

[π is generically isomorphism by Lemma 2]   unitary for $\lambda, \mu \in \mathbb{R}$



submodule Vir-generated by 1

$\Downarrow$ unitary

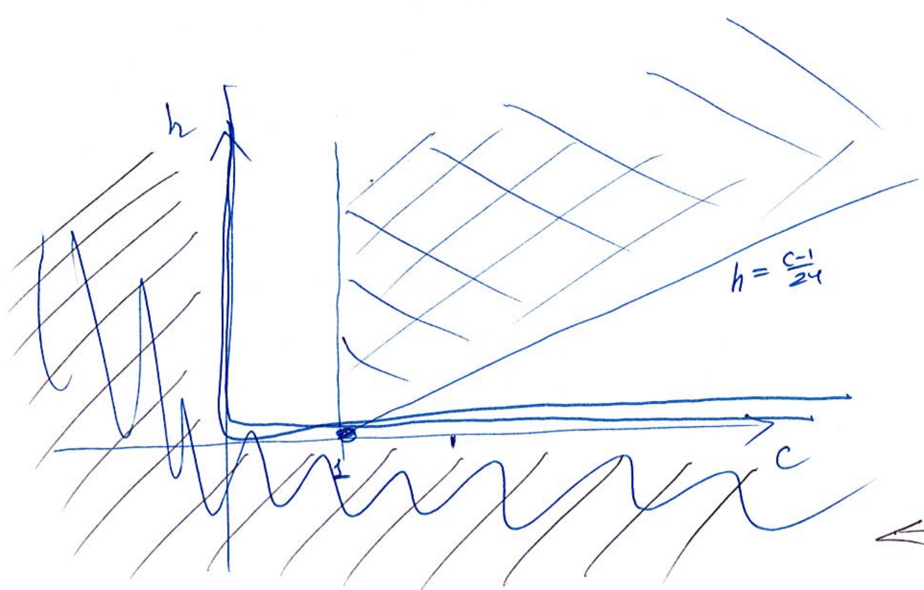
$\boxed{\text{it is irreducible}}$

$\Rightarrow$ $\boxed{\text{h. weight Vir-module } L_{h,c} \text{ with } c = 1+12\lambda^2,\ h = \frac{\lambda^2+\mu^2}{12},\ \lambda,\mu\in\mathbb{R} \text{ is always unitary}}$

$c = 1+12\lambda^2 \geq 1$ & $h = \frac{\lambda^2+\mu^2}{2} = \frac{\mu^2}{2} + \frac{c-1}{24} \geq \frac{c-1}{24}$

$\boxed{\text{Corollary 3}}$: If $\underset{\mathbb{R}}{c} \geq 1$, $\underset{\mathbb{R}}{h} \geq \frac{c-1}{24}$ $\Rightarrow$ $\boxed{L_{h,c} - \text{unitary}}$

irreducible Vir-module.

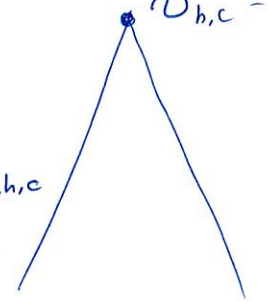


$\leftarrow$ bad area as we'll see next

Q: When $L_{h,c}$ is unitary?

Lemma 3: A necessary condition for $L_{h,c}$ to be unitary is $h, c \in \mathbb{R}_{\geq 0}$.

▶



$L_{h,c}$

$v_{h,c}$ - high. weight vector of $L_{h,c}$

$$(v_{h,c}, v_{h,c}) = 1$$

$$(L_{-n} v_{h,c}, L_{-n} v_{h,c}) = (\underbrace{L_{-n}^{+} L_{-n}}_{L_n L_{-n}} v_{h,c}, v_{h,c}) \ominus$$

$$L_{-n} L_n v_{h,c} = 0 \quad (n > 0)$$

$$\ominus \; (\underbrace{[L_n, L_{-n}]}_{2n L_0 + \frac{n^3-n}{12} K} v_{h,c}, v_{h,c}) = \boxed{2n \cdot h + \frac{n^3-n}{12} \cdot c}$$

Must be in $\mathbb{R}_{\geq 0}$ $\forall n$.

$\underline{n=1} \Rightarrow h \in \mathbb{R}_{\geq 0}$

$\underline{n \to \infty} \Rightarrow c \in \mathbb{R}_{\geq 0}$

■

Observation: If V_1, V_2 are two unitary $\mathfrak{g}$-repr-s, then so is $V_1 \otimes V_2$

with form $$\boxed{(\overset{V_1}{\overset{\Downarrow}{v_1}} \otimes w_1, \overset{V_2}{\overset{\Downarrow}{v_2}} \otimes w_2) := (v_1, v_2) \cdot (w_1, w_2)}$$

$$\left[\begin{aligned} (a(v_1 \otimes w_1), v_2 \otimes w_2) &= (a v_1, v_2) \cdot (w_1, w_2) + (v_1, v_2) \cdot (a w_1, w_2) \\ &= (v_1, a^+ v_2) \cdot (w_1, w_2) + (v_1, v_2) \cdot (w_1, a^+ w_2) = (v_1 \otimes w_1, a^+(v_2 \otimes w_2)) \end{aligned} \right]$$

In our setup: $L_{h,c}$ is unitary if $\left. \begin{matrix} c \geq 1 \\ h \geq \frac{c-1}{24} \end{matrix} \right\}$

$L_{0,1}$ is unitary $\left(\begin{matrix} c=1 \\ h=0 \end{matrix} \right)$

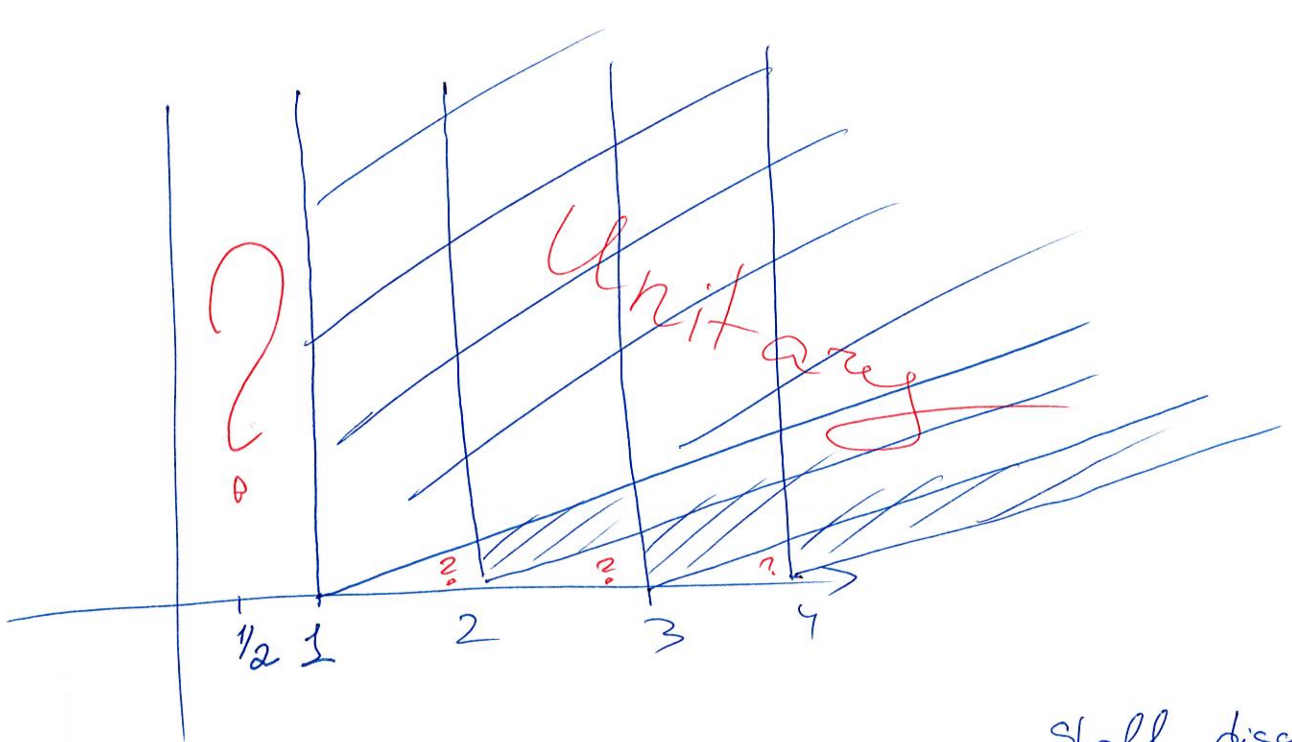
$\rightsquigarrow \underbrace{L_{0,1} \otimes L_{0,1} \otimes \ldots \otimes L_{0,1}}_{n-1} \otimes L_{h,c}$ – unitary (by Observation above).

$v_{v_{0,1} \otimes \ldots \otimes v_{0,1} \otimes v_{h,c}}$ – highest weight vector

as it's not irreducible $\rightsquigarrow$ look at submodule generated by it. (as it is unitary it must be irred)

$\rightsquigarrow$ get $\boxed{L_{h, c+(n-1)} \leftarrow \text{unitary.}}$!

Upshot



$\boxed{\text{Thm 1}}$: $L_{h,c}$ is unitary if $c \geq 1$, $h \geq 0$. ← Shall discuss in a few weeks.

Next time: Several unitary $L_{h, c=\frac{1}{2}}$

Mention later What happens at general $\boxed{\begin{matrix} 0 \leq c < 1 \\ h > 0 \end{matrix}}$