Lecture 8

02/11/2021

Last time:

- $Vir \ltimes \mathcal{A} \curvearrowright F_\mu \xrightarrow[\text{depending on } \lambda\in\mathbb{C}]{\text{1-param. deformation}} \boxed{Vir \curvearrowright F_\mu.}$ ($\tilde{L}_n$ from Lecture 7)

- $\lambda, \mu \in \mathbb{R} \Rightarrow$ this is a unitary repr-n.
(Note: Originally unitary for $\mathcal{A}$-action, but then also for Vir-action)

- $F_\mu \overset{\text{v.space}}{=} \mathbb{C}[x_1, x_2, \ldots]$, $1 \in F_\mu$

$$\left.\begin{aligned} K(1) &= 1+12\lambda^2 \\ L_0(1) &= \frac{\lambda^2+\mu^2}{2} \\ L_{>0}(1) &= 0 \end{aligned}\right\} \Rightarrow$$

Vir-module homomorphism

$$\boxed{M_{h,c} \longrightarrow F_\mu}, \qquad h = \frac{\lambda^2+\mu^2}{2}, \quad c = 1+12\lambda^2$$

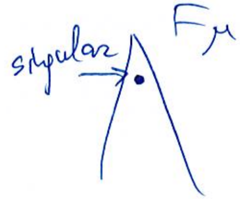
Vir-Verma

isom only for generic λ, μ!

Rmk 1: E.g. when $\lambda=0$ & $\mu=0$, then F_0 is actually not a simple Vir-module!

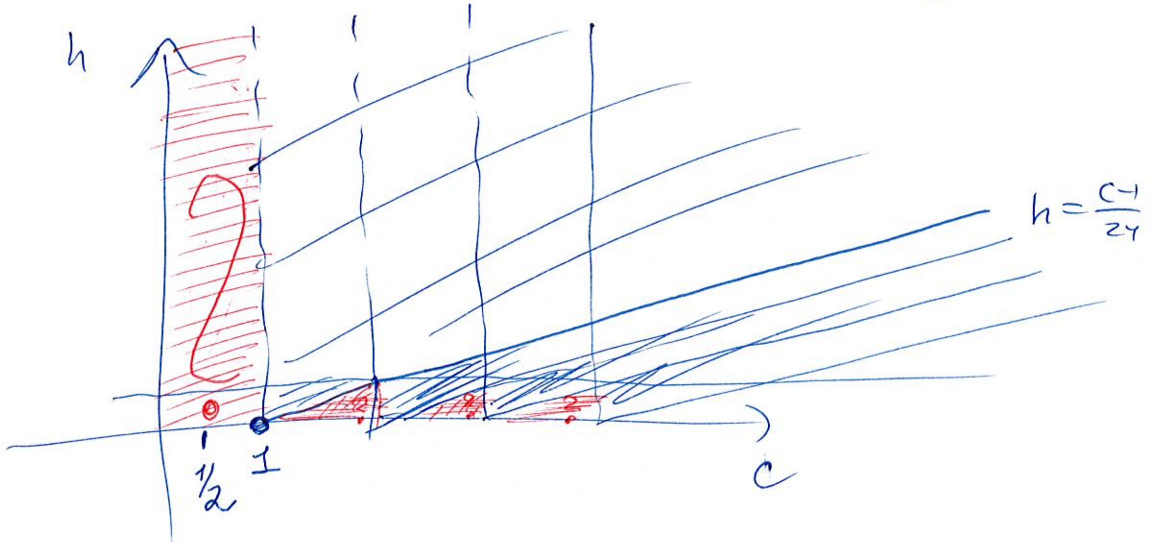
Indeed, consider $x_1 \in F_{\mu=0}$. Then:

$$\left.\begin{aligned} L_1(x_1) &= \sum_{j\geq 0} (a_{-j}\, a_{1+j})(x_1) = a_0 \underbrace{a_1(x_1)}_{1} = 0, \qquad a_1 = \partial/\partial x_1 \\ L_{>1}(x_1) &= 0 \quad \text{similarly} \end{aligned}\right\} \Rightarrow$$



$\Rightarrow F_{\mu=0} \neq$ irreducible.

- $L_{h,c}$ is unitary in the region below:



Know: Not unitary if $h \notin \mathbb{R}_{\geq 0}$ or $c \notin \mathbb{R}_{\geq 0}$.

> Thm: Unitary in all small ← triangles
>
> Thm: In the region $0 < c \leq 1$ the answer is more complicated, see p. 10

Rmk2: $\boxed{V, V' \text{ - unitary repr-s of } \mathfrak{g} \Rightarrow V \otimes V' \text{ - unitary repr. of } \mathfrak{g}}$

Last time applied: $L_{h,c} \otimes \underbrace{L_{1,0} \otimes L_{1,0} \otimes \cdots \otimes L_{1,0}}_{n-1}$ - unitary $\Rightarrow$ $\boxed{L_{h+(n-1),c} \text{ - unitary}}$

Look at the Vir-submodule gen-d by the $\otimes$product of highest weight vectors

But insted we could take

$$\boxed{L_{h_1,c_1} \otimes L_{h_2,c_2} \otimes \cdots \otimes L_{h_n,c_n} \text{ with } \begin{array}{l} c_k \geq 1 \\ h_k \geq \frac{c_k - 1}{24}. \\ (1 \leq k \leq n) \end{array}}$$

However: That wouldn't improve the above shadowed region!

Next: $c = \frac{1}{2}$ case to be discussed below!

Def: Let $\delta \in \{0, \frac{1}{2}\}$. Let C_δ be the $\mathbb{C}$-alg. gen-d by "fermions" $\{\psi_m \mid m \in \delta + \mathbb{Z}\}$ subject to:

$$\boxed{[\psi_m, \psi_n]_+ := \psi_m \psi_n + \psi_n \psi_m = \delta_m^{-n}}$$

(Just a Clifford algebra)

Case $\delta = 0$ ← "Ramond sector"

Case $\delta = \frac{1}{2}$ ← "Neveu - Schwarz sector".

This algebra C_δ acts on

$$\boxed{V_\delta := \underbrace{\Lambda^\bullet}_{\text{exterior algebra}: \ \xi_m \xi_n = -\xi_n \xi_m} \{\xi_n \mid n \geq 0, n \in \delta + \mathbb{Z}\} \curvearrowleft C_\delta}$$

via

$n > 0$

$\psi_{-n} \mapsto$ left multiplication by ξ_n

$\psi_n \mapsto \partial / \partial \xi_n$

$\psi_0 \longmapsto \frac{1}{\sqrt{2}} \left(\xi_0 + \partial / \partial \xi_0 \right)$

← should be perceived as a super-analogue of the previous $A \curvearrowright F_\mu$

Basis of V_δ is: $\left\{ \underbrace{\xi_{i_1} \wedge \xi_{i_2} \wedge \dots}_{\text{finitely many!}} \;\middle|\; \begin{array}{l} 0 \le i_1 < i_2 < \dots \\ i_k \in \delta + \mathbb{Z} \end{array} \right\}$

("finite wedges")

1) $\xi_n \cdot (\xi_{i_1} \wedge \xi_{i_2} \wedge \dots) = \xi_n \wedge \xi_{i_1} \wedge \xi_{i_2} \wedge \dots \xi_{i_k} \wedge \xi_{i_{k+1}} \wedge \dots = \begin{cases} 0, & \text{if } i_k = n \\ (-1)^k \cdot \xi_{i_1} \wedge \xi_{i_2} \wedge \dots \wedge \xi_{i_k} \wedge \xi_n \wedge \xi_{i_{k+1}} \wedge \dots \end{cases}$

getting $\pm$ sign pick k s.t. $i_k \le n < i_{k+1}$

2) $\frac{\partial}{\partial \xi_n} (\xi_{i_1} \wedge \xi_{i_2} \wedge \dots \wedge \xi_{i_k} \wedge \xi_{i_{k+1}} \wedge \dots) = \begin{cases} 0, & \text{if } i_k < n \\ (-1)^{k-1} \xi_{i_1} \wedge \dots \wedge \xi_{i_{k-1}} \wedge \xi_{i_{k+1}} \wedge \dots, & \text{if } i_k = n \end{cases}$

Pick k s.t. $i_k \le n < i_{k+1}$

Verification of the action is straightforward

Exercise!

Sketch

$n, m > 0$

- $\xi_n \xi_m + \xi_m \xi_n = 0 \iff [\psi_n, \psi_m]_+ = 0$.
- $\frac{\partial}{\partial \xi_n}\frac{\partial}{\partial \xi_m} + \frac{\partial}{\partial \xi_m}\frac{\partial}{\partial \xi_n} = 0 \iff [\psi_{-n}, \psi_{-m}]_+ = 0$
- $\xi_{-n} \xi_m + \xi_m \xi_{-n}$: $\xi_{i_1} \wedge \xi_{i_2} \dots$
  - $n \ne m$: get zero
  - $n = m$: $\frac{\partial}{\partial \xi_n} \xi_n + \xi_n \frac{\partial}{\partial \xi_n}$: $\xi_{i_1} \wedge \xi_{i_2} \wedge \dots \wedge \xi_{i_k} \stackrel{\xi_n}{\vee} \wedge \xi_{i_{k+1}} \wedge \dots$

    $i_k \le n < i_{k+1}$

Case 1 : $i_k < n < i_{k+1}$

$\xi_n \frac{\partial}{\partial \xi_n}$ acts by 0

$\frac{\partial}{\partial \xi_n} \xi_n$ acts by 1

Case 2 : $i_k = n < i_{k+1}$

$\frac{\partial}{\partial \xi_n} \xi_n$ acts by 0

$\xi_n \, \partial/\partial \xi_n$ acts by 1

$\underrightarrow{C\delta} \curvearrowright V_\delta$

Clifford alg.

<u>Prop 1</u> (Hwk 4 problem): For $\delta \in \{0, \frac{1}{2}\}$ consider op-s $L_n : V_\delta \circlearrowleft$ via:

$$\boxed{L_n = \delta_{n,0} \cdot \frac{1-2\delta}{16} + \frac{1}{2} \sum_{j \in \delta + \mathbb{Z}} j \underbrace{:\psi_{-j} \psi_{j+n}:}}$$

normally ordered product

$$\boxed{:\psi_i \psi_j: = \begin{cases} \psi_i \psi_j, & \text{if } i \leq j \\ -\psi_j \psi_i, & \text{if } i > j \end{cases}}$$

satisfy:

1) $[\psi_m, L_n] = (m + \frac{n}{2}) \psi_{m+n}$

2) $[L_n, L_m] = (n-m) L_{n+m} + \delta_{n,-m} \cdot \frac{n^3 - n}{24}$

! <u>Upshot</u>: Get $\boxed{Vir \curvearrowright V_\delta}$ with K acting via $\boxed{c = \frac{1}{2}}$

$\boxed{\mathrm{Vir} \curvearrowright V_\delta \quad \text{where} \quad \delta = 0 \text{ or } \tfrac{1}{2}}$

Q: Is it irreducible?

A: No, for V_δ is naturally $\mathbb{Z}/2\mathbb{Z}$-graded by setting $\deg(\xi_i) = \bar{1} \in \mathbb{Z}/2\mathbb{Z}$, & action of Vir preserves this grading!

$$\Rightarrow \boxed{V_\delta = \underbrace{V_\delta^+}_{\substack{\text{even} \\ \text{part}}} \oplus \underbrace{V_\delta^-}_{\substack{\text{odd} \\ \text{part}}}} \qquad \leftarrow V_\delta^\pm \text{ - Vir subrepr-n of } V_\delta$$

Thm 1: For $\delta = 0, \frac{1}{2}$, both V_δ^+, V_δ^- are <u>irreducible</u>. $\leftarrow$ See p. 9 for an outline of the proof

i.e. $V_0 = \underline{V_0^+} \oplus \underline{V_0^-}$, $\quad V_{\frac{1}{2}} = \underline{V_{\frac{1}{2}}^+} \oplus \underline{V_{\frac{1}{2}}^-}$

all irreducible

$\boxed{\delta = 0}$ $\quad V_0 = \underbrace{V_0^+}_{1^{\psi}} \oplus \underbrace{V_0^-}_{\psi \xi_0}$ $\qquad V_0 = \Lambda^{\circ}\{\underline{\xi_0}, \xi_1, \xi_2, \dots\}$

$1 \in V_0^+$ – h. wt. vector of h. wt $\underline{\left(h = \frac{1}{16}, c = \frac{1}{2}\right)}$

$\xi_0 \in V_0^-$ – h. wt vector $\left(\begin{array}{l} \text{as } L_{>0}(\xi_0) = 0 \\ L_0(\xi_0) = \frac{1}{16}\xi_0 \end{array}\right) \Rightarrow$ also of h. wt $\underline{\underline{\left(\frac{1}{16}, \frac{1}{2}\right)}}$.

Modulo Thm 1 get: $\boxed{V_0^+ \simeq V_0^- \simeq L_{\frac{1}{16}, \frac{1}{2}}}$

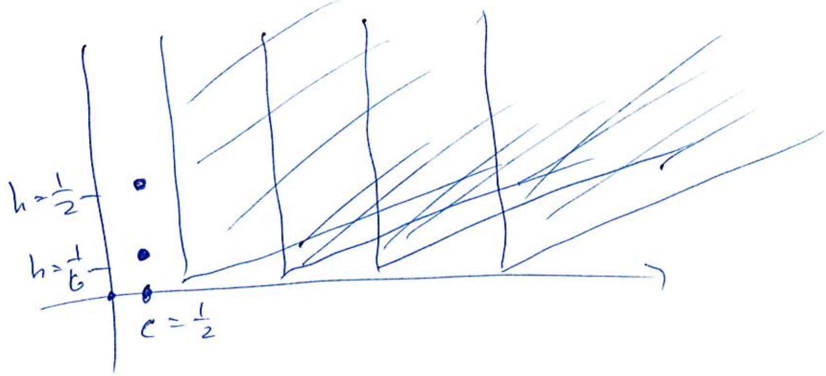
$\boxed{\delta = \frac{1}{2}}$ $\quad V_{\frac{1}{2}} = V_{\frac{1}{2}}^+ \oplus V_{\frac{1}{2}}^-$ $\qquad V_{\frac{1}{2}} = \Lambda^{\circ}\{\underline{\xi_{\frac{1}{2}}}, \xi_{\frac{3}{2}}, \xi_{\frac{5}{2}}, \dots\}$

$1 \in V_{\frac{1}{2}}^+$ – h. wt. vector of h. wt $\underline{(0, \frac{1}{2})}$

$\xi_{\frac{1}{2}} \in V_{\frac{1}{2}}^-$ – h. wt vector $\left(L_{\geq 1}(\xi_{\frac{1}{2}}) = 0\right)$ of h. wt $\underline{\left(h = \frac{1}{2}, c = \frac{1}{2}\right)}$

$$\left(\begin{array}{l} L_0(\xi_{\frac{1}{2}}) = \underbrace{\frac{1}{2}\sum_{j \in \frac{1}{2}+\mathbb{Z}} j :\psi_{-j}\psi_j:}(\xi_{\frac{1}{2}}) \\ = \sum_{j \in \frac{1}{2}+\mathbb{Z}_{\geq 0}} j\, \psi_{-j}\underline{\psi_j(\xi_{\frac{1}{2}})} \overset{j=\frac{1}{2}}{=} \frac{1}{2}\xi_{\frac{1}{2}} \end{array} \right)$$

Modulo Thm 1: $\boxed{\begin{array}{l} V_{\frac{1}{2}}^+ \simeq L_{0, \frac{1}{2}} \\ V_{\frac{1}{2}}^- \simeq L_{\frac{1}{2}, \frac{1}{2}} \end{array}}$



Lemma 1: V_δ has a unitary form determined by

$$\boxed{\begin{array}{c}(\xi_{i_1}\wedge\dots\wedge\xi_{i_k},\ \xi_{j_1}\wedge\dots\wedge\xi_{j_l}) = \delta_{i_1 j_1}\cdots\delta_{i_k j_l}\cdot\delta_{k,l} \\ i_1<\dots<i_k \qquad j_1<\dots<j_l.\end{array}}$$

i.e. finite wedges – orthonormal basis

and:

$$(\psi_m(v), w) = (v, \psi_{-m}(w)) \quad \longleftarrow \text{ obvious}$$

Check at home:

$$(L_m(v), w) = (v, L_{-m}(w)) \quad \longleftarrow \text{ follows from f-la expressing } L_m \text{ via } \psi\text{'s.}$$

$\boxed{\text{Corollary}}$: V_δ – unitary Vir-module $\Longrightarrow$ $V_\delta^{\pm}$ – unitary Vir-module

above discussion $\Longrightarrow$ $\boxed{L_{0,\frac{1}{2}},\ L_{\frac{1}{16},\frac{1}{2}},\ L_{\frac{1}{2},\frac{1}{2}} \text{ – unitary}}$

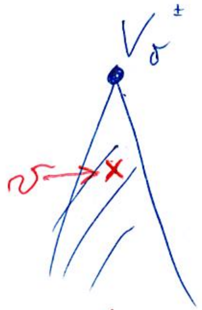
We'll see much later that

Claim: $\boxed{L_{h,\ c=\frac{1}{2}} \text{ - unitary} \iff h \in \{0, \tfrac{1}{16}, \tfrac{1}{2}\}}$

! Using this, we can immediately prove Thm 1.

$V_\delta = V_\delta^+ \oplus V_\delta^-$, $\delta = 0, \frac{1}{2}$.

If $V_\delta^\pm$ is not irreducible, they must have non-trivial <u>singular</u> vectors (killed by $L_{>0}$)



$K(v) = \frac{1}{2} \cdot v$

$L_0(v) = (\delta + \overset{\mathbb{Z}_{>0}}{\Downarrow}{k}) \cdot v$

b/c $[L_0, L_{-n}] = n L_{-n}$

$\Rightarrow$ Take a Vir-subrep-n generated by $v \in \text{Sing}(V)$

$\Downarrow$

it's h. wt & irreducible (b/c of unitarity)

$\Downarrow$

$L_{\delta+k, \frac{1}{2}}$ - unitary $\Rightarrow$ ↯

Thm: (1) For $c = 1 - \frac{6}{(n+2)(n+3)}$, $n \in \mathbb{Z}_{\geq 0}$, there are only finitely many h s.t.

$$\boxed{L_{h,c} \text{ - unitary}}$$

$\left(\text{e.g. } n=1 \Rightarrow c = \frac{1}{2} \Downarrow \text{ possible } h\text{'s}: 0, \frac{1}{16}, \frac{1}{2} \right)$

(2) For $0 < c < 1$ not of the above form,

$$\boxed{L_{h,c} \text{ - not unitary } \forall h}$$

We shall prove some parts of this result much closer to the end of the term!

Q: What about characters, i.e. generating series of dim-s of graded pieces?

$$V_\delta = V_\delta^+ \oplus V_\delta^- \qquad\qquad Vir \curvearrowright V$$

$$\boxed{\mathrm{ch}\, V_\delta^\pm \ - ?} \qquad\qquad \boxed{\mathrm{ch}\, V := \sum_{\lambda\in\mathbb{C}} q^\lambda \dim(\underbrace{V[\lambda]}_{\text{generalized eigenspace for } L_0\text{-action with eigenvalue } \lambda})}$$

$$\underline{\underline{\delta=0}}: \ \underbrace{V_0^+ \oplus V_0^-}_{\simeq V_0^+ \oplus V_0} \simeq V_0 \quad \Rightarrow \quad \mathrm{ch}\, V_0^\pm = \frac{1}{2}\, \underbrace{\mathrm{ch}\, V_0}_{\underbrace{(1+q^0)}_{2}(1+q^1)(1+q^2)\cdots} = \boxed{(1+q)(1+q^2)(1+q^3)\cdots}$$

compare to Fock f-la

$$\frac{1}{(1-q)(1-q^2)\cdots} = \mathrm{ch}\, F_0$$

(Using $V_0 = \Lambda^\circ\{\xi_0, \xi_1, \xi_2, \ldots\}$ with a basis consisting of finite wedges)

$$\underline{\underline{\delta=\tfrac{1}{2}}}: \ \mathrm{ch}(V_{1/2}^+ \oplus V_{1/2}^-) \overset{\mathrm{ch}}{=} (V_{1/2}) = q^{1/16}(1+q^{1/2})(1+q^{3/2})(1+q^{5/2})\cdots$$

$$\left(V_{\frac12} = \Lambda^\circ \{\xi_{\frac12}, \xi_{\frac32}, \xi_{\frac52}, \ldots\} \right.$$

$$\left. \underline{\text{Recall}}: L_0 = \frac{1}{16} + \frac{1}{2}\sum_j j :\ldots: \right)$$

$\mathrm{ch}\, V_{1/2}^+$, $\mathrm{ch}\, V_{1/2}^-$ are obtained by looking at $q^{\mathbb{Z}}$ or $q^{\frac12+\mathbb{Z}}$-parts above

For the rest of today, let's recast various equalities involving a_n, L_n via "quantum fields"

V - v. space $\rightsquigarrow$ $V[z]$, $V[z,z^{-1}]$, $V((z)) = \left\{ \sum_{i\in\mathbb{Z}} v_i z^i \mid v_i = 0 \ \forall i \ll 0 \right\}$

$$V[[z,z^{-1}]] = \left\{ \sum_{i\in\mathbb{Z}} v_i z^i \mid v_i \in V \right\}.$$

If V - algebra $\Rightarrow$ $V[z]$, $V[z,z^{-1}]$, $V((z))$ - algebras

$V[[z,z^{-1}]]$ - not an algebra!

Warning: $\left(\sum_{k=-\infty}^{+\infty} z^k \right) \cdot (1-z) = 0 \Rightarrow$ it has zero divisors!

! Note: We have a well-defined linear operator $\frac{d}{dz}$ on all of the above 4 spaces!

Def: "Delta-function" $\delta(w,z)$ is:

$$\boxed{\delta(w-z) = \sum_{n\in\mathbb{Z}} z^{-n-1} w^n}$$

Key Properties

$$(z-w)\,\delta(w-z) = 0$$

$$\frac{1}{2\pi i}\oint_{|z|=1} \delta(w-z) f(z)\,dz = f(w) \quad \forall \text{ Laurent pol-l } f$$

For our purposes, let's consider:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in \mathcal{U}(\mathcal{A})_{/(\mathbb{1}-1)}[[z,z^{-1}]]$$

Exercise: Verify that all rel-s on $\{a_n\}$ are equiv. to:

$$\boxed{a(z)a(w) = :a(z)a(w): + \frac{1}{(z-w)^2}}$$

extended in the region $|z| \gg |w|$.

! ZOOM got disrupted due to internet issues ⇝ see here instead!