Lecture #9

This Week: gl_∞, $\mathfrak{a}_\infty$, $\bar{\mathfrak{a}}_\infty$ ⟵ will be sitting inside — Old friends: A, Vir.

02/23/2021

Def 1: gl_∞ - the Lie algebra of matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with only finitely many nonzero entries

$$[A,B] = AB - BA. \leftarrow \text{usual Lie bracket}$$

- gl_∞ has a basis $\{E_{ij}\}_{i,j\in\mathbb{Z}}$ s.t. $[E_{ij}, E_{kl}] = \delta_{jk}\cdot E_{il} - \delta_{il}\cdot E_{kj}$

- Similar to $gl_n \curvearrowright (\,) = \mathbb{C}^n$

We have $gl_\infty \curvearrowright (\,) = V \longleftarrow V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}v_j$ ← basis

Multiplication of an ∞-size matrix by an ∞-size column

$V \circlearrowleft gl_\infty$ ⟵ $\boxed{E_{ij}(v_k) = \delta_{jk}\cdot v_i}$

- $\boxed{gl_\infty \curvearrowright V \rightsquigarrow gl_\infty \curvearrowright S^mV, \Lambda^m V \quad \forall m\in\mathbb{Z}}$

- gl_∞ - $\mathbb{Z}$-graded Lie algebra
  $\deg(E_{ij}) = j-i$

(Note: V - $\mathbb{Z}$-graded module via $\deg(v_k) = -k$. Indeed $\deg(E_{ij}) + \deg(v_j) = \deg(v_i)$)

$\Downarrow$

Triangular Decomp: $gl_\infty = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

$\mathfrak{n}_-$ ↑ lower-Δ; $\mathfrak{h}$ ← diagonal; $\mathfrak{n}_+$ ← upper triangular matrices

- $\forall \lambda \in \mathfrak{h}^* \rightsquigarrow \underline{M_\lambda^\pm}, \; L_\lambda^\pm$ $\qquad L_\lambda^\pm = M_\lambda^\pm / \mathrm{Ker}(\cdot,\cdot)_\lambda$

h.wt modules — Verma — irreducibles.

- $\dagger : gl_\infty \circlearrowleft$ $\quad E_{ij}^\dagger = E_{ji}$, i.e. the antilinear anti-involution $\dagger$ is just the transposition.

- $gl_\infty \curvearrowright \underline{V}$ - unitary (with $\{v_k\}_{k\in\mathbb{Z}}$ being an orthonormal basis)

[<u>Verification</u>: $(E_{ij} v_k, v_l) \overset{?}{=} (v_k, E_{ij}^\dagger v_l)$
$\quad \delta_{jk}\cdot\delta_{il} = \delta_{il}\cdot\delta_{jk}$]

$\Downarrow$ <u>Cor</u>: All $S^m V$, $\Lambda^m V$ are unitary gl_∞-modules

---

<u>However</u>: V, $S^m V$, $\Lambda^m V$ — not! highest weight modules!

This stays in contrast to finite case: $\quad gl_n \curvearrowright \mathbb{C}^n$ where $n_+ \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = 0$

But now for $gl_\infty \curvearrowright \begin{pmatrix}\vdots\\\vdots\\\vdots\end{pmatrix}$ <u>NO vector</u> is killed by n_+-action!

! <u>Hwk Problem</u>: V, $S^m V$, $\Lambda^m V$ — irreducible gl_∞-modules

Key Goal: To find some appropriate analogue of V, $S^m V$, $\underline{\underline{\Lambda^m V}}$ in the h.wt. category

A: We shall see this can be done for exterior powers, using semi-infinite wedge construction

Def 2: a) An elementary $\frac{\infty}{2}$-wedge is a formal infinite wedge product $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots$ with $i_0 > i_1 > i_2 > \dots$ s.t. $i_{k+1} = i_k - 1$ $\forall k \gg 0$

b) The semi-infinite wedge space $\Lambda^{\frac{\infty}{2}} V$ is the $\mathbb{C}$-vector space with basis given by expressions in a).

- $$\boxed{\Lambda^{\frac{\infty}{2}} V = \bigoplus_{m \in \mathbb{Z}} \underbrace{\Lambda^{\frac{\infty}{2}, m} V}_{\operatorname{span}\{V_{i_0} \wedge V_{i_1} \wedge V_{i_2} \wedge \dots \,|\, i_k = m - k \text{ for } k \gg 0\}}}$$ ← Splitting w.r.t. behavior of i_k as $k \to \infty$

examples: $V_1 \wedge V_0 \wedge V_{-1} \wedge V_{-2} \wedge V_{-3} \wedge \dots \in \Lambda^{\frac{\infty}{2}, 1} V$

$V_5 \wedge V_3 \wedge V_0 \wedge V_{-2} \wedge V_{-3} \wedge \dots \notin$

Key actor for today: $\Lambda^{\frac{\infty}{2}, m} V$

Prop 1: There is a natural action ($\forall m \in \mathbb{Z}$)

$$\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2}, m} V$$

via Leibniz rule

$$\underset{\mathfrak{gl}_\infty}{a} \left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots \right) = a(v_{i_0}) \wedge v_{i_1} \wedge v_{i_2} \wedge \dots$$
$$+ v_{i_0} \wedge a(v_{i_1}) \wedge v_{i_2} \wedge \dots$$
$$+ v_{i_0} \wedge v_{i_1} \wedge a(v_{i_2}) \wedge \dots$$
$$+ \dots$$

(Note: In exterior world

$v_i \wedge v_j = -v_j \wedge v_i$

$v_i \wedge v_i = 0$

$\Downarrow$

we reorder each summand so that indices strictly decrease)

! $\boxed{\text{Exercise}}$ (Hwk 5)

Idea: $a = E_{ij}$ $\longrightarrow$ $v_{i_0} \wedge v_{i_1} \wedge \dots \wedge v_{i_{k-1}} \wedge \overbrace{E_{ij}(v_{i_k})}^{\delta_{j i_k} \cdot v_i} \wedge v_{i_{k+1}} \wedge \dots$

$E_{ij}(v_{i_k}) = 0$ for $k >> 0$.

---

eg: $E_{31}(v_5 \wedge v_4 \wedge v_2 \wedge \underline{\underline{v_1}} \wedge v_0 \wedge v_{-1} \dots) = v_5 \wedge v_4 \wedge v_2 \wedge \underbrace{v_3}_{E_{31}(v_1)} \wedge v_0 \wedge v_{-1} \wedge \dots$

$= - v_5 \wedge v_4 \wedge v_3 \wedge v_2 \wedge v_0 \wedge v_{-1} \wedge \dots$

• $gl_\infty \curvearrowright \Lambda^{\frac{\infty}{2}} V \rightsquigarrow \boxed{gl_\infty \curvearrowright \underbrace{\Lambda^{\frac{\infty}{2}, m} V}_{\ni\ \psi_m := v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots}}$

• $\boxed{\text{Claim}}$: ψ_m is a highest wt vector

• $E_{ij}(v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots) \overset{j>m}{=\!=} 0$, $i<j$

$\overset{j\le m}{=\!=} \underbrace{v_m \wedge v_{m-1} \wedge \dots \wedge v_{j+1} \wedge \overset{E_{ij}(v_j)}{v_i} \wedge \underbrace{v_{j-1} \wedge v_{j-2} \wedge \dots}_{\ni\ v_i}}_{=0}$ (as v_i appears twice)

• $E_{ii}(\psi_m) = \begin{cases} 0, & \text{if } i>m \\ \psi_m, & \text{if } i\le m \end{cases}$

Def 3: Define a $\mathbb{Z}$-grading on $\Lambda^{\frac{\infty}{2},m}V$ via

$$\Lambda^{\frac{\infty}{2},m}V = \bigoplus_{d\le 0} \Lambda^{\frac{\infty}{2},m}V[d], \text{ where}$$

$$V^{\frac{\infty}{2},m}V[d] = \text{span}\left\{ v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots \;\middle|\; \begin{array}{l} i_0 > i_1 > i_2 > \dots \\ i_k = m-k \quad \forall k \gg 0 \\ \sum_{k\ge 0} (i_k + k - m) = -d \end{array} \right\}$$

← Note the "−" sign

So: $\boxed{\deg(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots) = -\sum_{k\ge 0} (i_k - (m-k))}$ ← finite sum due to stability condition $i_k - (m-k) = 0$ for $k \gg 0$

Note: $\underline{\dim \Lambda^{\frac{\infty}{2},m} V[d] = p(-d)}$, $d \le 0$ ← number of partitions of size $-d$

Rmk: The "−" sign above is needed for $\Lambda^{\frac{\infty}{2},m}V$ to be a $\mathbb{Z}$-graded rep-n of $\mathbb{Z}$-graded gl_∞

- $\underbrace{\Lambda^{\frac{\infty}{2}, m} V}_{\Psi_m}$

$$\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \cdots$$

Given any partition $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n > 0$ $\quad (\nu_1 + \cdots + \nu_n = |\nu|)$

$$\Downarrow$$

$$v_{m+\nu_1} \wedge v_{m-1+\nu_2} \wedge \cdots \quad \in \Lambda^{\frac{\infty}{2}, m} V[-|\nu|]$$

Conclusion: $\quad \mathfrak{gl}_\infty \curvearrowright \underbrace{\Lambda^{\frac{\infty}{2}, m} V}_{\Psi}$ $\quad$ w.r.t. basis $\{E_{ii}\}$ of $\mathfrak{h}$.

Easy: ψ_m generates entire $\Lambda^{\frac{\infty}{2}, m} V$ $\quad \psi_m$ - h.wt. vector with h. wt $\boxed{\omega_m = (\ldots, 1, 1, 0, 0, \ldots)}$ ($i-1$, i, $i+1$)

$$E_{ii}(\psi_m) = E_{ii}(v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \cdots) = \begin{cases} 0, & \text{if } i > m \\ v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \cdots & \text{if } i \leq m \end{cases}$$

Note: ω_m is infinite in both directions!

Prop 2: $\forall m \in \mathbb{Z}$, $\Lambda^{\frac{\infty}{2}, m} V$ is an irreducible h. wt. (already know) repn L_{ω_m} of $\mathfrak{gl}_\infty$, which is moreover unitary.

▶ Last time: h. wt. module + unitary str $\Rightarrow$ it is irreducible.

So: it suffices to show that $\Lambda^{\frac{\infty}{2}, m} V$ is a unitary $\mathfrak{gl}_\infty$-module.

Claim: The unitary form on $\Lambda^{\frac{\infty}{2}, m} V$ with the basis of $\frac{\infty}{2}$-wedges forming an orthonormal basis is actually unitary w.r.t. $\mathfrak{gl}_\infty$-action.

Easy Exercise!

Need to check: $\left(\underset{E_{ij}}{a} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots), v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \cdots \right) = \left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots, \underset{E_{ji}}{a^\dagger} (v_{j_0} \wedge v_{j_1} \wedge \cdots) \right)$

$v_j \qquad v_i$

Cor: If h.wt. $\lambda = (\lambda_i)_{i \in \mathbb{Z}}$ is s.t. $\lambda_i \in \mathbb{R}$ and $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ and zero for $|i| \gg 0$, then L_λ - unitary (here, L_λ is the irred. h.wt. λ repr-n of $\mathfrak{gl}_\infty$)

- First of all, such λ $\underbrace{\text{can be written}}_{\text{easy exercise!}}$ as $\beta + \sum_j n_j \omega_j$ (finite sum; $n_j \in \mathbb{Z}_{\geq 0}$)

Here: $\omega_j = (\ldots 1, 1, \ldots \underset{j}{1} | \underset{j+1}{0}, 0, \ldots)$ & $\underbrace{(\ldots, \beta, \beta, \beta, \ldots)}_{\text{constant sequence}}$

- $L_{\omega_j} \overset{\text{Prop2}}{\simeq} \Lambda^{\frac{\infty}{2},j} V$ - irreducible & unitary

- $L_\beta \leftarrow$ 1-dim $\simeq \mathbb{C}$ with E_{ij} acts trivially for $j \neq i$

  E_{ij} — // — $\beta \cdot Id$ for $j = i$.

  unitary for $\beta \in \mathbb{R}$ (<u>obvious!</u>)

$\Downarrow$

- $L_\beta \otimes \prod_j L_{\omega_j}^{\otimes n_j}$ — not necessarily irred., but if we look at submodule U gen-d by $\underbrace{v_\beta^+ \otimes \prod (v_{\omega_j}^+)^{\otimes n_j}}_{\text{h. wt. vector}}$ } U-h. wt. submodule which is unitary

  (L_β unitary, L_{ω_j} unitary)

it must be irreducible : $L_{\beta + \sum n_j \omega_j} = \underline{\lambda} \simeq U$

$\Rightarrow$ $\boxed{L_{\underline{\lambda}} \text{ - unitary!}}$

The opposite is also true (under minor assumptions) as shown in the next result.

<u>**Prop 3**</u>: If an irreducible $\mathfrak{gl}_\infty$-repr. L_λ is unitary

$\lambda = (\ldots \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots)$

$$\left.\begin{array}{l} \text{and } \lambda_i = \lambda_+ \in \mathbb{R}\ (\forall i \gg 0) \\ \lambda_i = \lambda_- \in \mathbb{R}\ (\forall i \ll 0) \end{array}\right\} \Rightarrow \text{then } \lambda \text{ satisfies conditions of previous Prop.}$$

i.e. $\boxed{\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}}$

- <u>Idea</u>: Reduce to $\mathfrak{sl}_2$-case.

- $\forall i: \mathfrak{sl}_2^{(i)} \subset \mathfrak{gl}_\infty$

$\mathfrak{sl}_2^{(i)} = \langle E_{i,i+1}, E_{i+1,i}, E_{ii} - E_{i+1,i+1} \rangle.$

$$\begin{tikzcd} \mathfrak{sl}_2 \arrow[r, "\dagger"] \arrow[d, hook] & \mathfrak{sl}_2 \arrow[d, hook] \\ \mathfrak{gl}_\infty \arrow[r, "\dagger"] & \mathfrak{gl}_\infty \end{tikzcd} \quad \leftarrow \text{compatibility}$$

- L_λ being unitary over $\mathfrak{gl}_\infty \Rightarrow L_\lambda$ unitary over $\mathfrak{sl}_2^{(i)}$

$v_\lambda^+ \in L_\lambda$ h.wt vector of $\mathfrak{gl}_\infty$ w.h. wt $\lambda = (\ldots, \lambda_{-2}, \lambda_{-1}, \lambda_0, \ldots)$

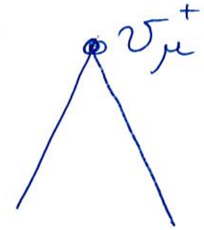
$$E_{ii}(v_\lambda^+) = \lambda_i \cdot v_\lambda^+$$

v_λ^+ – h.wt. vector w.r.t. $\mathfrak{sl}_2^{(i)}$ of h.wt. $\underbrace{\lambda_i - \lambda_{i+1}}$, as $(E_{ii} - E_{i+1,i+1})(v_\lambda^+) = (\lambda_i - \lambda_{i+1}) v_\lambda^+$

$\Downarrow$

Look at $\mathfrak{sl}_2^{(i)}$-submodule generated by v_λ^+, which is unitary (as the submodule of L_λ.)

• sl_2-setup: L_μ $\mu \in \mathbb{C}$
irr. of sl_2.



Q: When L_μ is unitary?

$(v_\mu^+, v_\mu^+) = 1.$

$\Downarrow$

$(f^n v_\mu^+, f^n v_\mu^+) = n! \cdot \mu \cdot (\mu-1) \cdot (\mu-2) \cdot \ldots \cdot (\mu-n+1) \notin \mathbb{R}_{\geq 0}$ for $n \gg 1$

unless $\mu \in \mathbb{Z}_{\geq 0}$.
(in the latter case $f^n v_\mu^+ = 0$ for $n > \mu$)

If $\mu \notin \mathbb{Z}_{\geq 0} \Rightarrow L_\mu$ - not unitary
$\mu \in \mathbb{Z}_{\geq 0} \Rightarrow L_\mu$ - unitary.
$\Rightarrow$ $\boxed{\text{A: } sl_2\text{-repr. } L_\mu \text{ is unitary iff } \mu \in \mathbb{Z}_{\geq 0}}$

So: Applying to the previous setup we conclude that

$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$. $\forall i$

Conclusion: Under mild conditions, we see that

$$L_\lambda\text{-unitary} \iff \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \quad \forall i.$$

L_λ: irred. h.wt. λ gl_∞-repr.

---

* **New Object**: $\overline{a}_\infty$ ← The point is that gl_∞ is too small to see interesting structures. We shall also see A, Vir only after extending our considerations from gl_∞ to $\overline{a}_\infty$ (& its central exten.)

Def 4: $\overline{a}_\infty$ is the Lie alg. of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with only finitely many nonzero diagonals (i.e. $a_{ij} = 0$ if $|i-j| \gg 1$) with the standard Lie bracket. $[A,B] = \underbrace{A\circ B}\,-\,\underbrace{B\circ A}$ ← each is in $\overline{a}_\infty$.

- $gl_\infty \subset \overline{a}_\infty \ni$ Identity
  - gl_∞: countable dimension
  - $\overline{a}_\infty$: uncountable dimension

$$\sum_j E_{j,j+1} = \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ & 0 & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

$$\sum_j E_{jj} = \begin{pmatrix} \ddots & & & \\ & 1 & & 0 \\ & & 1 & \\ & 0 & & \ddots \end{pmatrix}$$

Rmks: 1) $\overline{\mathfrak{a}}_\infty$ - $\mathbb{Z}$-graded

$\overline{\mathfrak{a}}_\infty = \bigoplus \overline{\mathfrak{a}}_\infty^i$ ← matrices with zeros outside of i^{th} diagonal.

Triangular decomp: $\overline{\mathfrak{a}}_\infty = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \longrightarrow M_\lambda^\pm, L_\lambda^\pm$ - as always.

2) $A \in \overline{\mathfrak{a}}_\infty,\ B \in \mathfrak{gl}_\infty \Rightarrow \underbrace{AB,\ BA \in \mathfrak{gl}_\infty}_{\text{Easy Exercise!}} \Rightarrow [A,B] \in \mathfrak{gl}_\infty$

Rmk: $\overline{\mathfrak{a}}_\infty$ can be viewed as an algebra of difference operators

i.e. formal sums $\underbrace{\sum_{k=\alpha}^{\beta} \gamma_k(n)\, T^k}$ $=: A$.

T - "shift operator"

$V = \{$basis of v_j's, $j \in \mathbb{Z}\}$, $T: V \circlearrowleft$ $\quad v_j \mapsto v_{j-1}$.

$\gamma_k : \mathbb{Z} \longrightarrow \mathbb{C}$

$$A(v_j) = \sum_k \gamma_k(j-k)\, v_{j-k}$$

$A =$ 

$= \bar{V}$.

$\boxed{Q}$: $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2}, m}\bar{V}$

$\rightsquigarrow$

Does it extend to $\bar{\mathfrak{a}}_\infty \curvearrowright \Lambda^{\frac{\infty}{2}, m}\bar{V}$?

We shall answer this q-n in the beginning of the next class...