Lecture # 10

02/25/2021

Last time: • $\boxed{\mathfrak{gl}_\infty \curvearrowright V,\ S^m V,\ \Lambda^m V}$

$\mathfrak{gl}_\infty$: $\mathbb{Z}$-graded via $\deg(E_{ij}) = j-i$

V: $\mathbb{Z}$-graded via $\deg(v_k) = \underbrace{-}_{}k$ — Note the "$-$" sign.

$$\underbrace{E_{ij}}_{j-i}\,(\underbrace{v_k}_{-k}) = \delta_{jk}\cdot \underbrace{v_i}_{-i} \qquad \text{deg: } j-i,\ -k,\ -i$$

Key obstacle: they are not h. wt!

• $\boxed{\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V}$

$\Lambda^{\frac{\infty}{2},m} V$: $\left\{ \text{basis: } v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge \dots \quad \begin{matrix} i_0 > i_1 > i_2 > \dots \\ i_k = m-k \text{ for } k \gg 0. \end{matrix} \right\}$

! **Explicitly**: $E_{rs}(v_{i_0}\wedge v_{i_1}\wedge \dots) = \begin{cases} 0, & \text{if } s \notin \{i_0, i_1, i_2, \dots\} \\ \underbrace{v_{i_0}\wedge v_{i_1}\wedge \dots \wedge v_{i_{k-1}} \wedge \underline{v_r} \wedge v_{i_{k+1}}\wedge \dots}_{\text{reorder to satisfy monotonous cond-n.}} & s = i_k \end{cases}$

• $\Lambda^{\frac{\infty}{2},m} V$ is $\mathbb{Z}$-graded via

$$\boxed{\deg(v_{i_0}\wedge v_{i_1}\wedge \dots) = \underset{\text{Note the sign}}{-}\sum_{k\geq 0} (i_k - (m-k)) \in \mathbb{Z}_{\leq 0}.}$$

• $\overset{\text{unitary}}{\Lambda^{\frac{\infty}{2},m} V} \ni \psi_m = v_m \wedge v_{m-1}\wedge v_{m-2}\wedge \dots$ ← h. wt. vector w. h.t. weight $\omega_m = (\dots 1, \underset{m}{1}, \underset{m+1}{0}, 0, \dots)$

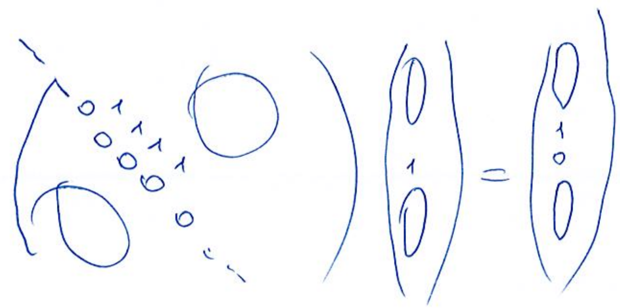
! ψ_m generates the entire $\Lambda^{\frac{\infty}{2},m} V$.

indeed: $\underbrace{E_{i_1,m-1} E_{i_0,m}}_{\text{fin product}}(\psi_m) = v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge \dots$

$\Longrightarrow \boxed{\Lambda^{\frac{\infty}{2},m} V \simeq \underbrace{L_{\omega_m}}_{\text{irr. } \mathfrak{gl}_\infty\text{-mod.}}}$

• $\overline{\mathfrak{a}}_\infty = \left\{ \underbrace{(a_{ij})_{i,j\in\mathbb{Z}}} \mid a_{ij}=0 \text{ if } |i-j|>N \text{ for some } N \right\}$

• $T = \sum_{i\in\mathbb{Z}} E_{i,i+1} \in \overline{\mathfrak{a}}_\infty$ with $\underline{T(v_k)=v_{k-1}}$.



$T^k = \sum_{i\in\mathbb{Z}} E_{i,i+k} \in \overline{\mathfrak{a}}_\infty$

↑ has 1 on the k-th diagonal.

• $\boxed{Q}$ $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V$

$\rightsquigarrow$

$\overline{\mathfrak{a}}_\infty \overset{?}{\dashrightarrow} \Lambda^{\frac{\infty}{2},m} V$

$\cup$

$A = \sum a_{ij} E_{ij}$ apply to $v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge \dots$ $\qquad v_i \rightsquigarrow v_{i+k}$

Let's split $A = \sum_k A_k$, $\boxed{A_k = \sum_{i\in\mathbb{Z}} a_{i,i+k} E_{i,i+k}}$

↑ has nonzero terms only on the k^{th} diagonal

• if $k \neq 0$, the action is well-defined

• if $k=0$: $\sum a_i E_{ii}(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\dots) = \Big(\underbrace{\sum_k a_{i_k}}_{\infty\text{-sum}}\Big)\cdot v_{i_0}\wedge v_{i_1}\wedge\dots$

not well-defined!

Fix: $\hat{\rho}: \left(\sum_{k\geq 0} a_{i_k} - \sum_{i\leq 0} a_i\right)$ ← finite sum.

We'll fix by a "standard regularization technique"

Explicitly: define

$$\boxed{\begin{aligned}&\hat\rho(E_{ij}) = \begin{cases}\rho(E_{ij}) - \mathbb{1}, & \text{if } i=j\le 0\\ \rho(E_{ij}), & \text{otherwise.}\end{cases}\\[1em] &\rho : gl_\infty \to \mathrm{End}(\Lambda^{\frac{\infty}{2},m} V)\end{aligned}}$$

Even though $\{E_{ij}\}$ - not a basis of $\bar{\mathfrak{a}}_\infty$, but we define linear map

$$\boxed{\begin{aligned}\hat\rho : \bar{\mathfrak{a}}_\infty &\xrightarrow{\text{linear map}} \mathrm{End}(\Lambda^{\frac{\infty}{2},m} V)\\ \Psi\quad&\\ A=(a_{ij})_{i,j\in\mathbb{Z}} &\longmapsto \sum a_{ij}\,\hat\rho(E_{ij})\end{aligned}}$$

$\boxed{\text{Easy Exercise}}$: $\hat\rho(A)$ is well-defined. $\leftarrow$ basically follows from the previous discussion

!! BUT: $\hat\rho$ is NOT a Lie alg. homom!

The discrepancy is measured $\underbrace{[\hat\rho(A), \hat\rho(B)] - \hat\rho([A,B])}_{\alpha(A,B)\in \mathrm{End}(\Lambda^{\frac{\infty}{2},m}V)}$ $\forall A,B\in\bar{\mathfrak{a}}_\infty$.

$A, B \in \mathfrak{gl}_\infty \xrightarrow[\text{4 quarters}]{\text{present via}} A = \left(\begin{array}{c|c} \boxed{A_{11}} & A_{12} \\ \hline A_{21} & \boxed{A_{22}} \end{array}\right)$ & $B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$

rows & columns $\in \mathbb{Z}_{\leq 0}$ rows & columns $\in \mathbb{Z}_{>0}$.

! **Note**: A_{12}, A_{21}, B_{12}, B_{21} have only fin. many nonzero entries

$\boxed{\boxed{\text{Prop 1}}}$: $$\boxed{\alpha(A, B) = \mathrm{Tr}\left(A_{12}B_{21} - B_{12}A_{21}\right) \cdot Id_{\Lambda^{\infty/2, m} V}}$$

Exercise (Hwk 5)

E.g. $A = E_{ij}$, $B = E_{k\ell}$ $\rightsquigarrow$ $\boxed{\alpha(E_{ij}, E_{k\ell}) = \begin{cases} 1, & \text{if } i \leq 0 < j \text{ \& } k = j, \ell = i \\ -1, & \text{if } i > 0 \geq j \text{ \& } -\text{''}- \\ 0, & \text{otherwise.} \end{cases}}$

More generally

General Explicit Formula:

$$\boxed{\alpha\big(\underbrace{(a_{ij})}_{A}, \underbrace{(b_{ij})}_{B}\big) = \sum_{i \leq 0 < j} a_{ij} b_{ji} - \sum_{j \leq 0 < i} a_{ij} b_{ji}}$$

$\boxed{\text{Lemma 1}}$: The bilinear map $\alpha : \overline{\mathfrak{gl}}_\infty \times \overline{\mathfrak{gl}}_\infty \longrightarrow \mathbb{C}$ defined by f-la in Prop 1 is a 2-cocycle on $\overline{\mathfrak{gl}}_\infty$, which is <u>not trivial</u>.

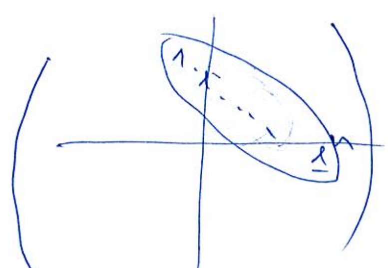
$\boxed{\text{"Japanese" 2-cocycle}}$

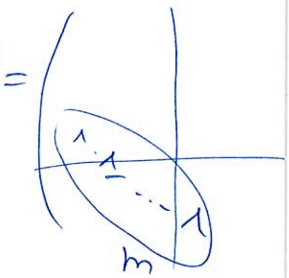
- The fact that α is a 2-cocycle comes immediately from the original defn: $\boxed{\alpha(A,B) = [\hat{\rho}(A), \hat{\rho}(B)] - \hat{\rho}([A,B])}$

$\boxed{\text{Easy Check}}$

k^{th} diagonal.

- Consider the el-s $\{T^k\}_{k\in\mathbb{Z}} \subset \overline{\mathfrak{gl}}_\infty$.

$T^n =$ 

$T^m =$ 

Easy Check : $\underline{\alpha(T^n, T^m) = n \cdot \delta_{n,-m}}$

Note: 1) $\{T^k\}_{k\in\mathbb{Z}}$ - pairwise commute in $\overline{\mathfrak{gl}}_\infty$ $(T^n \cdot T^m = T^{n+m} = T^m \cdot T^n)$

2) $\alpha\big|_{\text{Lie subalg. gen-d by } \{T^k\}}$ = same as the one used to define $\underbrace{\mathcal{A}}_{\text{Heisenberg alg.}}$

If α was 2-coboundary $\Rightarrow$ its restriction to any Lie subalg. would be 2-coboundary; BUT it's not the case for $\mathcal{A}$!

Rmk: However $\alpha|_{gl_\infty \times gl_\infty}$ is actually a 2-coboundary!

! Important feature of $\mathfrak{a}_\infty$ vs gl_∞

$A, B \in gl_\infty$

$$\alpha(A,B) = \mathrm{Tr}\big(J \circ \underbrace{[A,B]}\big)$$

$$J = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right) \qquad [A,B] = \left(\begin{array}{c|c} & * \\ \hline * & * \end{array}\right)$$

$$[A_{11}, B_{11}] + A_{12}B_{21} - B_{12}A_{21}$$

$\Downarrow$

α – 2-coboundary

(as it's given by evaluation of a functional $X \mapsto \mathrm{Tr}(J \cdot X)$ on $X = [A,B]$)

Def: Let $\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}\cdot K$ be the central 1-dim extension via 2-cocycle α.

← Key Object of Interest

$\boxed{\text{Thm 1}}$: Let $\hat{\rho}: \mathfrak{a}_\infty \longrightarrow \mathrm{End}(\Lambda^{\frac{\infty}{2},m} V)$ be the lin. map

$$K \longmapsto \mathrm{Id}$$

$$\bar{\mathfrak{a}}_\infty \ni A \longmapsto \hat{\rho}(A) \leftarrow \text{defined before}$$

is a Lie alg. homomorphism! $\rightsquigarrow$ $\boxed{\mathfrak{a}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V}$

Proof: Follows from the previous discussions. ∎

Rmk: 1) $\mathfrak{a}_\infty$ is $\mathbb{Z}$-graded, with $\deg(K)=0$

2) $\Lambda^{\frac{\infty}{2},m}V$ is a $\mathbb{Z}$-graded repr-n of $\mathfrak{a}_\infty$

3) $$\begin{array}{ccc} \overset{\text{Heisenberg}}{\mathcal{A}} & \hookrightarrow & \mathfrak{a}_\infty \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ a_k & \longmapsto & T^k \\ K & \longmapsto & K \end{array} \Bigg\} \rightsquigarrow \boxed{\mathcal{A} \curvearrowright \Lambda^{\frac{\infty}{2},m}V}$$

$\bar{\omega}_m \in (\mathfrak{a}_\infty[0])^*$ defined via

$$\begin{cases} K \longmapsto 1 \\ \sum a_i E_{ii} \longmapsto \begin{cases} \sum_{j=1}^m a_j, & \text{if } m \geq 0 \\ -\sum_{j=m+1}^0 a_j, & \text{if } m<0 \end{cases} \end{cases}$$

h.wt. vector is
$\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$

$\boxed{\text{Prop 2}}$: $\Lambda^{\frac{\infty}{2},m}V$ is the irreducible h.wt. repr-n $L_{\bar{\omega}_m}$ of $\mathfrak{a}_\infty$

Moreover, it is unitary.

(note $K^\dagger = K$)

- ψ_m is killed by $\mathfrak{n}^+$ (deg>0 part of $\mathfrak{a}_\infty$)
- $\sum a_i E_{ii} (v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots) = \begin{cases} a_m + a_{m-1} + \ldots + a_1 + 0 + 0 + 0 + \ldots & m>0 \\ 0 + 0 + \ldots \; 0, & m=0 \\ \downarrow \text{next page}, & m<0 \end{cases}$

$$\sum_{i\in\mathbb{Z}} a_i E_{ii} \Big(\underbrace{v_m}_{} \wedge \underbrace{v_{m-1}}_{} \wedge \underbrace{v_{m-2}}_{} \wedge \cdots \Big) \xRightarrow{\text{regularization}} \underbrace{\sum_{k>0} a_{i_k}}_{\cancel{\quad} \; a_m + a_{m-1} + a_{m-2} + \cdots} - \sum_{i\le 0} a_i =$$

negative

$$= -a_0 - a_{-1} - \cdots - a_{m+1}$$

h.wt + unitarity $\Rightarrow$ irreducible

$\mathcal{A} \hookrightarrow \mathfrak{a}_\infty$ – above.

Also! $Vir \hookrightarrow \mathfrak{a}_\infty$ in many ways!

basis $\{v_k\}$

[Hwk 1, Problem 4] $\rightsquigarrow$ $W \curvearrowright V_{\gamma,\beta} = \{ g(t)\, t^\gamma (dt)^\beta \mid g(t) \in \mathbb{C}[t, t^{-1}] \}$

Witt alg

make a change $k \leftrightarrow -k$, i.e. $v_k \leftrightarrow v_{-k}$

explicitly given $\boxed{L_n(v_k) = (k - \gamma - (n+1)\beta)\, v_{k-n} \quad \forall\, k, n \in \mathbb{Z}}$

identify v_k of $V_{\gamma,\beta}$ with v_k of V

$$\boxed{L_n \longmapsto \sum_{k\in\mathbb{Z}} (k - \gamma - (n+1)\beta)\, E_{k-n,k} \in \overline{\mathfrak{a}}_\infty}$$

Thus we get a Lie alg. homom.

$$\boxed{\overline{\varphi}_{\gamma,\beta} : \begin{array}{ccc} \overline{W} & \hookrightarrow & \overline{\mathcal{O}}_{\infty} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ L_n & \longmapsto & L_n \end{array}}$$

We used 2-cocycle α to extend $\overline{\mathcal{O}}_\infty \rightsquigarrow \mathcal{O}_\infty$

So: need to know how α behaves on $\operatorname{Im} \overline{\varphi}_{\gamma,\beta}$.

Exercise (Hwk 5)

$$\boxed{\underbrace{\alpha}_{\text{Jap. 2-cocycle}}(L_n, L_m) = \delta_{n,-m}\left(\frac{n^3-n}{12}\, c_\beta + 2n \cdot h_{\gamma,\beta}\right)}$$

$c_\beta = -12\beta^2 + 12\beta - 2$

$h_{\gamma,\beta} = \dfrac{\gamma(\gamma + 2\beta - 1)}{2}$.

Prop 3: For any $\gamma, \beta \in \mathbb{C}$, there is a Lie alg. homom (injective if $c_\beta \neq 0$)

$$\boxed{\begin{aligned} \varphi_{\gamma,\beta} : \mathrm{Vir} &\longrightarrow \mathcal{O}_\infty \\ \mathbf{c} &\longmapsto c_\beta \cdot K \\ L_n &\longmapsto \overline{\varphi}_{\gamma,\beta}(L_n) \quad \text{if } n \neq 0 \\ L_0 &\longmapsto \overline{\varphi}_{\gamma,\beta}(L_0) + h_{\gamma,\beta} \cdot K \end{aligned}}$$

$\mathfrak{O}\mathcal{L}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V$

$\uparrow \varphi_{\gamma,\beta}$

Vir

$\rightsquigarrow$ $\boxed{\mathrm{Vir} \curvearrowright \underbrace{\Lambda^{\frac{\infty}{2},m} V}_{\ni\ \psi}}$

$\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots$

- ψ_m is clearly killed by L_n $(n>0)$
- $\mathbf{C}(\psi_m) = c_\beta \cdot \psi_m = \underbrace{(-12\beta^2 + 12\beta - 2)}_{\text{central charge}} \cdot \psi_m$
- $L_0(\psi_m) = \dfrac{(\gamma-m)(\gamma+2\beta-m-1)}{2} \cdot \psi_m$

Hwk 5 Exercise.

$\boxed{\text{Corollary}}$: Have a Vir - homom.

$$\boxed{M^+_\lambda \xrightarrow{\varphi^*_{\gamma,\beta}(\Lambda^{\frac{\infty}{2},m} V)} \Lambda^{\frac{\infty}{2},m} V_{\gamma,\beta}}$$

$\lambda = \left(\dfrac{(\gamma-m)(\gamma+2\beta-m-1)}{2},\ -12\beta^2 + 12\beta - 2\right)$

{Later: Generically isom.}

Back to Heisenberg alg:

$$\mathcal{A} \hookrightarrow \mathfrak{a}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V$$

$$\rightsquigarrow \boxed{\mathcal{A} \curvearrowright \Lambda^{\frac{\infty}{2},m} V.}$$

<u>Recall</u>: $\Lambda^{\frac{\infty}{2},m} V$ are $\mathbb{Z}_{\leq 0}$-graded with $\underbrace{\dim \Lambda^{\frac{\infty}{2},m} V[-d] \overset{\forall d \geq 0}{=} p(d)}$

partitions of d

$\psi_m \in \Lambda^{\frac{\infty}{2},m} V$

- $\to$ killed by upper-Δ matrices $\Rightarrow a_n(\psi_m) = 0 \quad \forall\, n > 0.$
- $\to a_0(\psi_m) = \mathrm{Id}(\psi_m) = m \cdot \psi_m$

$\boxed{\text{Conclusion}}$: Get an $\mathcal{A}$-homom. $\boxed{F_m \xrightarrow{\sigma_m} \Lambda^{\frac{\infty}{2},m} V, \quad 1 \longmapsto \psi_m}$

$\boxed{\text{Prop 4}}$: σ_m is an ~~in~~ isomorphism.

▸ Both modules are $\mathbb{Z}$-graded & have the same dimensions & F_m-irred

$\Rightarrow \sigma_m$ - isom. ☒