03/02/2021

Lecture # 11

Last time • $\mathfrak{gl}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V$

$\rightsquigarrow$

$\mathfrak{a}_\infty \curvearrowright \Lambda^{\frac{\infty}{2},m} V$ ← Key construction from last time

$\mathfrak{a}_\infty = \underbrace{\overline{\mathfrak{a}}_\infty}_{\mathbb{Z}\times\mathbb{Z}\text{-matrices with fin. many non-zero diagonals}} \oplus \mathbb{C}\cdot K$

- **Observation**: (1) $\mathcal{A} \hookrightarrow \mathfrak{a}_\infty$

$a_k \longmapsto T^k$ ← matrix with 1's on the k'th diagonal and 0's elsewhere

$K \longmapsto K$

(2) $Vir \xhookrightarrow{\varphi_{\gamma,\beta}} \mathfrak{a}_\infty$

- **Last result from Thur**:

$$\boxed{V^{\frac{\infty}{2},m} V \simeq F_m \text{ as } \mathcal{A}\text{-modules}}$$

($\simeq$ Verma module over A).

Fock

This shall be the starting point for today's lecture!

For today, let's introduce:

- $\mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2},m} V$ ← spanned by $\frac{\infty}{2}$-wedges with $i_k = m-k$ $\forall k \gg 1$.
  "fermionic space".

- $\mathcal{B}^{(m)} = F_m$ ← as v. space it's $\mathbb{C}[x_1, x_2, \dots]$
  "bosonic space"

- $\mathcal{F} := \bigoplus_{m\in\mathbb{Z}} \mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2}} V$ ← has basis $\left\{ v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots \;\middle|\; \begin{array}{l} i_0 > i_1 > i_2 > \dots \\ i_{k+1} = i_k - 1 \;\; \forall k \gg 1 \end{array} \right\}$

- $\mathcal{B} := \bigoplus_{m\in\mathbb{Z}} \mathcal{B}^{(m)} = \mathbb{C}[z, z^{-1}; x_1, x_2, \dots]$
  with $\mathcal{B}^{(m)}$ identified with $z^m \cdot \mathbb{C}[x_1, x_2, \dots]$

Know $\sigma_m : \mathcal{F}^{(m)} \xrightarrow[\sim]{\mathcal{A}\text{-module isom.}} \mathcal{B}^{(m)}$, $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots \longmapsto 1$ $\xrightarrow[\text{over all } m]{\text{Combine}}$

$\mathfrak{r}_\infty \circlearrowright$, $\mathcal{A} \circlearrowright$

$$\boxed{\sigma : \mathcal{F} \xrightarrow[\text{isom.}]{\mathcal{A}\text{-module}} \mathcal{B}}$$

← Key for today

"Boson – Fermion" correspondence

$\boxed{Q1}$: Which polynomials arise as the images of elementary $\frac{\infty}{2}$-wedges?

$\boxed{Q2}$: How to extend $\mathcal{A} \circlearrowright \mathcal{B}$ to $\mathfrak{r}_\infty \circlearrowright \mathcal{B}$?

Def 1: (a) For $i \in \mathbb{Z}$, define the wedging operator $\xi_i = \hat{v}_i : \mathcal{F} \longrightarrow \mathcal{F}$, $\quad \psi \longmapsto v_i \wedge \psi$

<u>Note</u>: $\hat{v}_i : \mathcal{F}^{(m)} \longrightarrow \mathcal{F}^{(m+1)}$.

(b) For $i \in \mathbb{Z}$, define the contracting operator $\xi_i^* = \check{v}_i : \mathcal{F} \longrightarrow \mathcal{F}$, $\quad \psi \longmapsto \check{v}_i(\psi)$

<u>Explicitly</u>:

$$\check{v}_i(\underbrace{v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots}_{i_0 > i_1 > i_2 > \cdots}) = \begin{cases} 0, & \text{if } i \notin \{i_0, i_1, \dots\} \\ (-1)^k \cdot v_{i_0} \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge \cancel{v_{i_k}} \wedge v_{i_{k+1}} \wedge \cdots, & \text{if } i_k = i. \end{cases}$$

<u>Note</u>: $\check{v}_i : \mathcal{F}^{(m)} \longrightarrow \mathcal{F}^{(m-1)}$

Lemma 1 (Exercise): For any $i, j \in \mathbb{Z}$, we have:

↑ Do it at home!

$$\boxed{\hat{v}_i \hat{v}_j + \hat{v}_j \hat{v}_i = 0, \quad \check{v}_i \check{v}_j + \check{v}_j \check{v}_i = 0, \quad \check{v}_i \hat{v}_j + \hat{v}_j \check{v}_i = \delta_{ij}}$$

$$gl_\infty \overset{\rho}{\curvearrowright} \Lambda^{\frac{\infty}{2}} V \qquad \rightsquigarrow \qquad \boxed{\rho(E_{ij}) = \xi_i \xi_j^*}$$

$E_{ij} \in gl_\infty$, $\quad v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots \in \Lambda^{\frac{\infty}{2}} V$

Recall that we had to modify $\rho \rightsquigarrow \hat{\rho}$ to extend to a_∞.

$$\boxed{\hat{\rho}(E_{ij}) = \begin{cases} \xi_i \xi_j^* - 1, & \text{if } i = j \le 0 \\ \xi_i \xi_j^*, & \text{elsewhere} \end{cases} \overset{def}{=} \; :\xi_i \xi_j^*:}$$

!

<u>Recall</u>:

$$\rho(E_{ij})(v_{i_0} \wedge v_{i_1} \wedge \cdots) = \begin{cases} 0, & \text{if } j \notin \{i_0, i_1, i_2, \dots\} \\ v_{i_0} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_i \wedge v_{i_{k+1}} \wedge \cdots & \text{if } i_k = j \end{cases}$$

$$a_k \longmapsto T^k = \sum_{i\in\mathbb{Z}} E_{i,i+k}$$
$$\mathcal{A} \longrightarrow \mathfrak{a}_\infty$$
$$\Rightarrow \boxed{\hat\rho(a_k) = \sum_{i\in\mathbb{Z}} \hat\rho(E_{i,i+k}) = \sum_{i\in\mathbb{Z}} :\xi_i \xi^*_{i+k}:}$$

gives the action $\mathcal{A} \curvearrowright \Lambda^{\frac{\infty}{2}} V = \mathcal{F}$.

$$a(z) = \sum_{k\in\mathbb{Z}} a_k z^{-k-1}$$

$$\boxed{\hat\rho(a(z)) = :\xi(z)\xi^*(z):}$$

Here, we set:

$$\xi(z) \stackrel{\text{def}}{=} \sum \xi_n z^{n-\frac{1}{2}}$$
$$\xi^*(z) \stackrel{\text{def}}{=} \sum \xi^*_n z^{-n-\frac{1}{2}}$$

Defn: Consider the following quantum fields

$$X(u) = \sum_{n\in\mathbb{Z}} \xi_n u^n \in \mathrm{End}(\mathcal{F})[[u,u^{-1}]]$$

$$X^*(u) = \sum_{n\in\mathbb{Z}} \xi^*_n u^{-n} \in \mathrm{End}(\mathcal{F})[[u,u^{-1}]]$$

$$\sigma : \mathcal{F} \xrightarrow{\sim} \mathcal{B}$$

$$\Gamma(u) = \sum \ldots = \sigma \circ X(u) \circ \sigma^{-1} \in \mathrm{End}(\mathcal{B})[[u,u^{-1}]]$$

$$\Gamma^*(u) = \sigma \circ X^*(u) \circ \sigma^{-1} \in \mathrm{End}(\mathcal{B})[[u,u^{-1}]].$$

i.e. $\Gamma(u)$, $\Gamma^*(u)$ are just $X(u)$, $X^*(u)$ "pulled through σ" to the $\mathcal{B}$-side

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Theorem 1 : For any $m \in \mathbb{Z}$, the operators

$$\Gamma(u) : \mathcal{B}^{(m)} \longrightarrow \mathcal{B}^{(m+1)}[[u, u^{-1}]]$$

$$\Gamma^*(u) : \mathcal{B}^{(m)} \longrightarrow \mathcal{B}^{(m-1)}[[u, u^{-1}]]$$

$$\mathcal{A} \curvearrowright \mathcal{B}^{(m)} \rightsquigarrow \mathcal{A} \curvearrowright \mathcal{B} = \bigoplus_m \mathcal{B}^{(m)} \cdot z^m$$

are explicitly given via the $\mathcal{A}$-action by:

$$\boxed{\begin{aligned} \Gamma(u) &= u^{m+1} \cdot z \cdot \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \\ \Gamma^*(u) &= u^{-m} \cdot z^{-1} \cdot \exp\left(-\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \end{aligned}}$$

Rmks: a) z, z^{-1} are in charge of $\mathcal{B}^{(m)} \longrightarrow \mathcal{B}^{(m\pm 1)}$.

b) $\{a_j\}_{j>0}$, $\{a_j\}_{j<0}$ commute among themselves $\Rightarrow \exp(A) = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$

c) For every $w \in \mathcal{B}^{(m)}$, each $u^?$-coeff. of the action of the right-hand sides on w is well-defined.

Instructive: Work this out for yourself at home.

Lemma 2 : For $j \in \mathbb{Z}$, we have :

$$\boxed{[a_j, \Gamma(u)] = u^j \cdot \Gamma(u) \quad , \quad [a_j, \Gamma^*(u)] = -u^j \cdot \Gamma^*(u)} \quad \underline{\underline{(1)}}$$

on the level of $\mathrm{End}(\mathcal{B})_{[[u,u^{-1}]]}$

recall: $\Gamma(u) = \sigma \circ X(u) \circ \sigma^{-1}$

recall: $\Gamma^*(u) = \sigma \circ X^*(u) \circ \sigma^{-1}$!

• Applying σ^{-1}, the first equality is equivalent:

$\sigma : \mathcal{F} \xrightarrow{\sim} \mathcal{B}$

$$\boxed{[T^j, X(u)] \overset{?}{=} u^j \cdot X(u).}$$

$$T^j = \sum_{i\in\mathbb{Z}} E_{i,i+j} \Rightarrow \hat{\rho}(T^j) = \sum_{i\in\mathbb{Z}} \hat{\rho}(E_{i,i+j}) = \sum_{i\in\mathbb{Z}} :\xi_i \xi^*_{i+j}:$$

$$\xRightarrow{\text{on } \mathcal{F} = \Lambda^{\frac{\infty}{2}} V} [\hat{\rho}(T_j), X(u)] \overset{\text{can ignore } :\cdots:}{=} \sum_{i\in\mathbb{Z}} [\xi_i \xi^*_{i+j}, X(u)] = \sum_{i,n} \underbrace{[\xi_i \xi^*_{i+j}, \xi_n]}\, u^n \;=$$

$$\left([\xi_i \xi^*_{i+j}, \xi_n] = \xi_i \xi^*_{i+j} \xi_n - \xi_n \xi_i \xi^*_{i+j} = \xi_i(-\xi_n \xi^*_{i+j} + \delta_{n,i+j}) - \xi_n \xi_i \xi^*_{i+j} \overset{\xi_i \xi_n + \xi_n \xi_i = 0}{=\!=} \delta_{n,i+j}\xi_i \right)$$

$$= \sum_{n\in\mathbb{Z}} \xi_{n-j} u^n = u^j X(u).$$ This proves $[T^j, X(u)] = u^j X(u)$, hence, 1st equality!

• 2nd equality $\longrightarrow$ do as Homework

■

You may need this in the problem 1 of Hwk 6.

Proof of Thm 1

Note: it's exactly the 2^{nd} exponent in $\Gamma(u)$ as appears in Thm 1.

Let $\Gamma_+(u) := \exp\left(-\sum_{i>0} \frac{a_i}{i} u^{-i}\right) \in \mathrm{End}(\mathcal{B})[\![u,u^{-1}]\!]$. Then:

$$\boxed{\begin{aligned} &\bullet\ [a_j, \Gamma_+(u)] = 0 \quad \forall j>0. \\ &\bullet\ [a_j, \underbrace{\Gamma_+(u)}] = u^j \cdot \Gamma_+(u) \quad \forall j<0 \end{aligned}} \quad (2)$$

$$\prod_{i>0} \exp\left(-\frac{a_i}{i} u^{-i}\right) \rightsquigarrow [a_j, \prod_{i>0} \exp\left(-\frac{a_i}{i}u^{-i}\right)] = \prod_{i \neq -j}^{i>0} \exp\left(-\frac{a_i}{i}u^{-i}\right) \times$$

$$\boxed{\begin{gathered} \underbrace{\times [a_j, \exp\left(+\frac{a_{-j}}{+j}\right) u^j]} \\ \exp\left(\frac{a_{-j}}{j} u^j\right) \cdot u^j. \end{gathered}}$$

Must do @ home !!! Easy Check $\Rightarrow$

Consider $\mathcal{B}^{(m)} \xrightarrow[\Gamma(u)\Gamma_+(u)^{-1}z^{-1}]{\Delta(u) \,=} \mathcal{B}^{(m)}[\![u,u^{-1}]\!]$

Note: z^{-1} is a map $\mathcal{B} \to \mathcal{B}$ sending $\mathcal{B}^{(m+1)} \to \mathcal{B}^{(m)}$

$$\boxed{[a_j, z] = \delta_{j,0} \cdot z.} \quad (3)$$

Combining (1, 2, 3), get the following f-la:

$$\boxed{[a_j, \Delta(u)] = \begin{cases} u^j \cdot \Delta(u), & j>0 \\ 0, & j \le 0 \end{cases}} \quad \underline{\underline{(4)}}$$

$\Rightarrow a_{-1}, a_{-2}, a_{-3} \dots$ commute with $\Delta(u)$.

$$\Downarrow$$

$$P(a_{-1}, a_{-2}, \dots) \underbrace{\Delta(u) \underset{B^{(m)}}{\mathbb{1}}} = \Delta(u) P(a_{-1}, a_{-2}, \dots) \mathbb{1}$$

Consider $\Delta(u) \cdot \mathbb{1}_m \in B^{(m)}[[u^{-1}, u]]$.

series in u with coeff-s being polynomials in $x_1, x_2, x_3 \dots$

$\underline{\underline{(4)}} \underset{\text{claim}}{\overset{\text{Easy}}{\Rightarrow}} \Delta(u) \cdot \mathbb{1}_m = \underbrace{\exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right)} \mathbb{1}_m \cdot \underbrace{F(u)}_{\text{series in } u, \text{ independent of } x\text{'s}} \Rightarrow$

Remarks: $\underline{\underline{F(u) = u^{m+1}}}$

$$\boxed{\Gamma(u) = z \cdot \underline{F(u)} \cdot \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(\sum_{j>0} -\frac{a_j}{j} u^{-j}\right)}$$

Remarks: recover $F(u)$. To this end:

$$\underset{B\text{-side}}{F(u)} = \underbrace{\langle \mathbb{1}^*_{m+1}, \Gamma(u) \underset{B^{(m)}}{\mathbb{1}_m} \rangle}_{\text{coeff. of } \underset{B^{(m+1)}}{\mathbb{1}_{m+1}} \text{ in } \Gamma(u)(\underset{B^{(m)}}{\mathbb{1}_m})} \xrightarrow[\mathcal{F}\text{-side}]{\text{Pull to}} \langle \psi^*_{m+1}, \overbrace{X(u)}^{\sum_{n \in \mathbb{Z}} \xi_n u^n} \psi_m \rangle = \underbrace{\langle \psi^*_{m+1}, \sum_{n \in \mathbb{Z}} \xi_n \psi_m \rangle u^n}$$

$\psi_m = v_m \wedge v_{m-1} \wedge \dots$

$u^{m+1} = 0 + \dots + 0 + u^{m+1} + 0 + \dots$

comes from $\underline{n = m+1}$.

$\Rightarrow \boxed{F(u) = u^{m+1}.}$ This proves the f-la for $\Gamma(u)$ from Thm 1!

*F-la for $\Gamma^*(u)$ — in the Homework 6

Now, we have f-las for $\Gamma(u)$, $\Gamma^*(u)$

$\boxed{\underline{\text{Q2}} \text{ was about extending } \mathcal{A} \curvearrowright \mathcal{B} \rightsquigarrow \mathfrak{a}_\infty \curvearrowright \mathcal{B}.}$

It suffices to write down f-las for the action of $E_{ij} \curvearrowright \mathcal{B}$

Corollary: Let $\boxed{\Gamma(u,v) := \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j}\right) \cdot \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_{j}\right)}$

(a) $$\rho\left(\sum_{i,j \in \mathbb{Z}} u^i v^{-j} E_{ij}\right) \underset{\text{on } \mathcal{B}^{(m)} \text{ or } \mathcal{F}^{(m)}}{=\!=\!=} \frac{(u/v)^m}{1 - v/u} \Gamma(u,v)$$

here: ρ is the action of $\mathfrak{gl}_\infty$

(b) $$\hat{\rho}\left(\sum_{i,j \in \mathbb{Z}} u^i v^{-j} E_{ij}\right) \underset{- \,\shortparallel\, -}{=\!=\!=} \frac{1}{1 - v/u} \left(\left(\frac{u}{v}\right)^m \Gamma(u,v) - 1\right).$$

here: $\hat{\rho}$ is the modified action of $\mathfrak{a}_\infty$

Proof of Corollary

- (a) $\Rightarrow$ (b) : $\hat{\rho}\left(\sum_{i,j} u^i v^{-j} E_{ij}\right) = \rho\left(\sum_{i,j} u^i v^{-j} E_{ij}\right) - \underbrace{\sum_{i\le 0} u^i v^{-i} \cdot \mathrm{Id}}_{\frac{1}{1-\frac{v}{u}}}$ – expanded in non-neg. powers of v.

- (a) $\rho(E_{ij}) = \xi_i \xi_j^* \Rightarrow \rho\left(\sum_{i,j} u^i v^{-j} E_{ij}\right) = X(u) X^*(v)$.

$$X(u) X^*(v) = u^{\overbrace{(m-1)+1}^{m}} \cdot \cancel{z} \cdot \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \overbrace{\exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)}^{\text{"2nd exp"}} \cdot v^{-m} \cdot \cancel{z^{-1}}$$

$$\cdot \underbrace{\exp\left(-\sum_{j>0} \frac{a_{-j}}{j} v^j\right)}_{\text{"3d exp"}} \exp\left(+\sum_{j>0} \frac{a_j}{j} v^{-j}\right)$$

Claim : (2nd exp) $\cdot$ (3d exp) = (3d exp) $\cdot$ (2nd exp) $\cdot$ factor $G(u,v)$

where $G(u,v) = \exp\left(\sum_{j>0} \frac{\cancel{j}}{j \cdot \cancel{j}} \cdot u^{-j} v^j\right) = \exp\left(\underbrace{\sum_{j>0} \frac{1}{j}\left(\frac{v}{u}\right)^j}_{-\log(1-\frac{v}{u})}\right)$

$= \frac{1}{1 - v/u}$

Important exercise to work out at home!