Lecture 12

Last time :

- $$\begin{array}{ccccc} & \bigoplus \mathcal{F}^{(m)} & & \bigoplus \mathcal{B}^{(m)} & \\ & \parallel & & \parallel & \\ \sigma : & \mathcal{F} & \xrightarrow{\ \sim\ } & \mathcal{B} & \\ & \parallel & & \parallel & \\ & \Lambda^{\frac{\infty}{2}} V & & \mathbb{C}[z, z^{-1}, x_1, x_2, \dots] & \end{array}$$

"boson-fermion correspondence"

$\mathcal{B}^{(m)} \longleftrightarrow z^m \cdot \mathbb{C}[x_1, x_2, \dots]$

- $\Gamma(z), \ \Gamma^*(z) \xleftarrow{\text{Thm1}}$ via $\exp(\dots a_j) \exp(\dots a_j)$

But a nice short way to write these f-las is:

$u^{m+1} \cdot z$ – for $\Gamma(z)$

$u^{-m} \cdot z^{-1}$ – for $\Gamma^*(z)$

& m keeps track of the component $\mathcal{B}^{(m)}$!

$$\boxed{\begin{aligned} \Gamma(z) &= uz \cdot \, :\exp\left(\int a(u)\, du\right): \\ \Gamma^*(z) &= z^{-1} \cdot \, :\exp\left(-\int a(u)\, du\right): \end{aligned}}$$

← here: $a(u) = \sum_{m \in \mathbb{Z}} a_m u^{-m-1}$

← For a_0: $\exp\left(\int \frac{a_0}{u}\, du\right) = \exp(a_0 \log u) = u^{a_0}$

a_0 acts on $\mathcal{B}^{(m)} = F_m$ exactly by $\underline{m}$!

- Two technical things:

(1) $\left[a_j, \exp\left(\frac{a_{-j}}{j} u^j\right)\right] = \exp\left(\frac{a_{-j}}{j} u^j\right) \cdot u^j$

(Direct Proof: $\forall \alpha, \beta \in$ Algebra, s.t. $[\alpha, \beta]$ commutes with β (i.e. $[[\alpha,\beta],\beta] = 0$) $\Rightarrow [\alpha, P(\beta)] = [\alpha, \beta] \cdot P'(\beta)$ $\forall$ power series P)

(2) $\exp(\lambda a_j)\exp(\mu a_{-j}) = \exp(\mu a_{-j})\exp(\lambda a_j)\cdot\exp(j\lambda\mu)$.

Argument : BCH (Baker–Campbell–Hausdorff)

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\ldots\ldots}$$ ← each next term involves further commutators of $[X,Y]$ with something else expressed via X, Y

$\Downarrow$

Lemma : If α, β satisfy $[\alpha,\beta]$ commutes with α & β, then:

$$e^\alpha e^\beta = e^\beta \cdot e^\alpha \cdot e^{[\alpha,\beta]}$$

►

$$\left.\begin{aligned} e^\alpha e^\beta &= e^{\alpha+\beta+\frac{1}{2}[\alpha,\beta]} \\ e^\beta e^\alpha &= e^{\beta+\alpha+\frac{1}{2}[\beta,\alpha]} \end{aligned}\right\} \Uparrow$$

Apply to $\alpha = \lambda a_j$, $\beta = \mu a_{-j}$ $\rightsquigarrow$ $[\alpha,\beta] = \lambda\mu j\cdot \underbrace{K}_{\substack{\text{acts by } 1 \\ \text{on each } \mathcal{B}^{(m)}}}$.

Today, we shall answer:

Q1: Describe the images of elementary $\frac{\infty}{2}$-wedges under $\mathcal{F} \xrightarrow[\sigma]{\sim} \mathcal{B}$

$v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots$

Def 1: For $k \in \mathbb{Z}_{\geq 0}$, define $S_k(X) \in \mathbb{C}[x_1, x_2, x_3, \dots]$ via

$X = \{x_1, x_2, x_3, \dots\}$

$$\boxed{\sum_{k \geq 0} S_k(X) \cdot z^k = \exp\left(\sum_{i \geq 1} x_i z^i\right)}$$

e.g. $S_2(X) = x_2 + \frac{x_1^2}{2}$

They are closely related to complete symmetric f-s $h_k(y)$, defined via

$y = \{y_1, y_2, \dots, y_N\}$

$$\boxed{h_k(y) = \sum_{\substack{p_i \geq 0 \\ p_1 + \dots + p_N = k}} y_1^{p_1} y_2^{p_2} \dots y_N^{p_N}}$$

$y = (y_1, \dots, y_N)$

Lemma 1: If $x_n = \dfrac{y_1^n + \dots + y_N^n}{n}$ $\forall n \geq 1$, then

$$\boxed{S_k(x) = h_k(y)}$$

$$\sum_{k\geq 0} S_k(x) z^k = \exp\Big(\sum_{n\geq 1} x_n z^n\Big) = \exp\Big(\sum_{n\geq 1} \frac{(y_1 z)^n + \dots + (y_N z)^n}{n}\Big) = \prod_{i=1}^{n} \exp\Big(\overbrace{\sum_{n\geq 1} \frac{(y_i z)^n}{n}}^{-\log(1-y_i z)}\Big)$$

$$= \prod_{i=1}^{n} \frac{1}{1-y_i z} = \sum_{k\geq 0} h_k(y) z^k \qquad \blacksquare$$

Def 2: To any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m)$, define (bosonic) Schur polynomials

$$\boxed{S_\lambda(x) = \det \begin{pmatrix} S_{\lambda_1}(x) & S_{\lambda_1+1}(x) & \dots & S_{\lambda_1+m-1}(x) \\ S_{\lambda_2-1}(x) & S_{\lambda_2}(x) & \dots & S_{\lambda_2+m-2}(x) \\ \dots & \dots & & \dots \\ S_{\lambda_m-m+1}(x) & \dots & \dots & S_{\lambda_m}(x) \end{pmatrix} = \det\big(S_{\lambda_i+j-i}(x)\big)_{i,j=1}^{m}}$$

Note: They are not symmetric (hence differ from classical symmetric Schur pol-s) but they are closely related to those – see next Remark

Rmk: 1) To get classical symmetric Schur polynomials,

replace $x_n = \dfrac{y_1^n + \cdots + y_N^n}{n}$ $\forall n$ (& apply Lemma 1)

(Usually: Symmetric Schur pol-s $(y_1, \ldots, y_N) = \det\left(h_{\lambda_i+j-i}(y)\right)$

Jacobi-Trudi identity.)

2) $S_{k<0}(x) = 0$ by default.

3) It is independent of $\lambda_m = 0$ or $\lambda_m \neq 0$.

$\Rightarrow S_\lambda(x)$ does not change if we add a few 0's at the end to λ.

4) Symmetric Schur pol-s $S_\lambda(y_1, \ldots, y_N)$ arise as characters

of irreducible $GL(N)$-representations.

Thm 1: For any $i_0 > i_1 > i_2 > \dots$ s.t. $i_k = -k \ \forall k \gg 0$, we have

$$\boxed{\sigma(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots) = \underbrace{S_\lambda(x)}_{\in \mathcal{B}^{(0)} = \mathbb{C}[x_1, x_2, x_3, \dots]} \quad \text{with } \lambda = (i_0, i_1+1, i_2+2, \dots)}$$

$$\sigma : \mathcal{F} \xrightarrow{\sim} \mathcal{B}.$$

Warning: In original f-las for $\mathcal{A} \curvearrowright \mathbb{C}[x_1, x_2, \dots]$
$\forall j > 0$: $a_j \mapsto j\, \partial/\partial x_j$
$a_{-j} \mapsto x_j$
$\rightsquigarrow$ **Now** $\boxed{\begin{array}{l} a_j \mapsto \partial/\partial x_j \\ a_{-j} \mapsto j x_j \end{array}}$

Let $\sigma(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots) =: P(x) \in \mathbb{C}[x_1, x_2, x_3, \dots]$

Pick another family of independent variables $y_1, y_2, y_3, \dots$

- $\langle 1, \underbrace{e^{y_1 a_1 + y_2 a_2 + \dots} P(x)}_{e^{y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + \dots} P(x)} \rangle = \langle 1, P(x+y) \rangle = P(y)$. $\longleftarrow$ $\mathcal{B}$-side

$x_1 + y_1,\ x_2 + y_2, \dots$

$\Big\| \ \mathcal{B}\text{-to-}\mathcal{F}$ $\qquad \wr\, \sigma$ $\mathcal{F}$-side

- $\langle \overset{v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots}{\overset{\|}{\psi_0}}, e^{y_1 T + y_2 T^2 + \dots} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots) \rangle = \langle \overset{v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots}{\overset{\|}{\psi_0}}, \underbrace{\left(\sum_{k \geq 0} S_k(y) T^k\right)}_{\begin{pmatrix} 1 & S_1(y) & S_2(y) & S_3(y) \\ & 1 & S_1(y) & S_2(y) & \dots \\ & & 1 & S_1(y) & \dots \\ & 0 & & 1 & \dots \end{pmatrix} \notin \mathfrak{a}_\infty} v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots \rangle$

$\overset{!}{\|}$ det (of submatrix)

recall: $T = \sum_{i \in \mathbb{Z}} E_{i,i+1}$

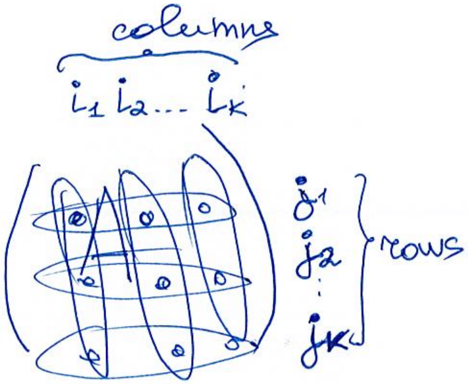
Exercise: Check it's well-defined given

• Toy Model

In fin. dim. case:

Given a lin. operator $A : V \longrightarrow W \rightsquigarrow \boxed{\Lambda^k A : \Lambda^k V \longrightarrow \Lambda^k W}$

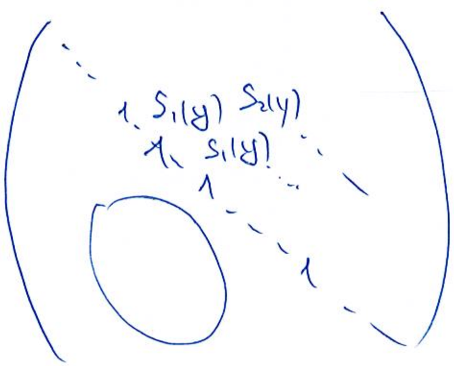
$\{v_i\}_{i=1}^{n}$ - basis, $\{w_j\}_{j=1}^{m}$ - basis



Then:

$$\boxed{\begin{gathered}\langle w_{j_1} \wedge w_{j_2} \dots \wedge w_{j_k},\ \Lambda^k A (v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k}) \rangle \\ \| \\ \det A^{j_1 j_2 \dots j_k}_{i_1 i_2 \dots i_k} \quad \longleftarrow \quad k \times k \text{ minor of matrix of } A \end{gathered}}$$

in our infinite setup



rows: $i_0, i_1, i_2, \dots$

columns: $0, -1, -2, \dots$

Check Details at home.

required f-la!

$i_k = -k \quad \forall k \gg 0$

$\Downarrow$

essentially compute det of a finite size matrix

$\mathcal{F}^{(0)} \xrightarrow{\sigma} \mathcal{B}^{(0)}$ – discussed above. But what about general m:

$$\boxed{\begin{array}{l} \mathcal{F}^{(m)} \longrightarrow \mathcal{B}^{(m)} = z^m\, \mathbb{C}[x_1, x_2, \ldots] \\ \| \\ \Lambda^{\frac{\infty}{2}, m} V \end{array}}$$

← Same question: what are the images of elementary wedges?

Likewise, you can show:

Cor: $\forall\ i_0 > i_1 > i_2 > \ldots$ s.t. $i_k = m - k\ \ \forall k \gg 0$:

$$\boxed{\begin{array}{c} \sigma(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) = z^m \cdot S_\lambda(x), \\ \lambda = (i_0 - m,\ i_1 - (m-1),\ i_2 - (m-2), \ldots) \end{array}}$$

just keeps track of $\mathcal{B}^{(m)}$

Upshot

$$\boxed{\begin{array}{ccc} \sigma : \mathcal{F} & \xrightarrow[\mathcal{A}\text{-homom.}]{\sim} & \mathcal{B} \\ \circlearrowleft \mathfrak{a}_\infty \supseteq \mathcal{A} & & \circlearrowleft \mathcal{A} \end{array}}$$

<u>Today</u>: Images of basis $\frac{\infty}{2}$-elementary wedges

<u>Tuesday</u>: $\mathfrak{a}_\infty$-action on $\mathcal{B}$-side

! <u>Rmk</u>: It's a particular feature of ∞-dim V.
(for $\dim V < \infty$: $S^k V$ - ∞-dim, $\Lambda^{k'} V$ - fin. dim.)

For the rest of today & next Tue, we'll do application to integrable systems:

- **KdV eq-n** (Korteweg–de Vries): $u = u(x,t)$

$$\boxed{u_t = \frac{3}{2} u \cdot u_x + \frac{1}{4} u_{xxx}}$$

Rmk: renormalizing x, t, u by scalars can change $\frac{3}{2}, \frac{1}{4}$ to any other constants!

- **KP eq-n.** (Kadomtsev–Petviashvili) $u = u(x,y,t)$

$$\boxed{u_{yy} = \left(u_t - \frac{3}{2} u \cdot u_x - \frac{1}{4} u_{xxx}\right)_x}$$

Goal: Construct a family of solutions using ∞-dim Lie algebras

Key tool: Infinite Grassmannian.

For the rest of today: focus on fin. dim. first.

- V - fin. dim. space $/\mathbb{C}$

Pick a basis $\{v_1, \ldots, v_n\}$ of $V \simeq \mathbb{C}^n$

$$\rightsquigarrow GL(V) \curvearrowright V \rightsquigarrow GL(V) \overset{\forall k}{\curvearrowright} \underbrace{\Lambda^k V}_{\text{irreducible with h.wt vector } v_1 \wedge v_2 \wedge \ldots \wedge v_k} \qquad 0 \leq k \leq n$$

here: $g(v_1 \wedge v_2 \wedge \ldots \wedge v_k) = g(v_1) \wedge g(v_2) \wedge \ldots \wedge g(v_k)$

Def 3: Let $\boxed{\Omega := GL(V)(v_1 \wedge v_2 \wedge \ldots \wedge v_k)}$

Lemma 2 (obvious): $\Omega = \{\text{all decomposable non-zero wedges}\}$

$$= \{x_1 \wedge \ldots \wedge x_k \in \Lambda^k V \mid x_1, \ldots, x_k - \text{lin. indep.}\}.$$

Closely related is:

Def 4: The k-Grassmannian, $Gr(k, V)$, is the set of k-dim subspaces in V.

Recall: $Gr(k,V)$ - projective variety via a Plücker embedding:

$$\boxed{\begin{array}{ccc} Pl: Gr(k,V) & \hookrightarrow & \mathbb{P}(\Lambda^k V) \\ \Psi & & \Psi \\ k\text{-dim subspace } W & \longmapsto & \text{projection of} \\ \text{pick its basis } w_1, \ldots, w_k & & w_1 \wedge w_2 \wedge \ldots \wedge w_k \\ & & (\text{Note: basis independent}) \end{array}}$$

$$\boxed{Gr(k,V) \simeq \Omega / \mathbb{C}^\times}$$

Clear

Following Lecture 11, define wedging & contraction operators:

* $\boxed{\hat{v} : \Lambda^k V \longrightarrow \Lambda^{k+1} V}$

$\forall v \in V$ $\quad v_{i_1} \wedge \dots \wedge v_{i_k} \longmapsto v \wedge v_{i_1} \wedge \dots \wedge v_{i_k}$.

* $\check{v}_i : \Lambda^k V \longrightarrow \Lambda^{k-1} V$ ← similar to Lecture 11. But: can do it basis-independent:

$\forall f \in V^*$, we have $\boxed{\check{f} : \Lambda^k V \longrightarrow \Lambda^{k-1} V}$

$$\begin{aligned} f^{\vee}(v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k}) = \; & f(v_{i_1}) \cdot v_{i_2} \wedge v_{i_3} \wedge \dots \\ & - f(v_{i_2}) \cdot v_{i_1} \wedge v_{i_3} \wedge v_{i_4} \wedge \dots \\ & + f(v_{i_3}) \cdot v_{i_1} \wedge v_{i_2} \wedge v_{i_4} \wedge \dots \\ & - \dots \end{aligned}$$

Def 5: For any $0 \le k \le n$, define the linear operator

Key Construction:

$$\boxed{\begin{gathered} S : \Lambda^k V \otimes \Lambda^k V \longrightarrow \Lambda^{k+1} V \otimes \Lambda^{k-1} V \\ S = \sum_{i=1}^{n} \hat{v}_i \otimes \check{v}_i \end{gathered}}$$

← $\boxed{\text{Exercise}}$: It's indep. of the basis $\{v_i\}_{i=1}^n$ of V.

Thm 2: For $\tau \in \Lambda^k V \setminus \{0\}$, we have:

$$\boxed{\tau \in \Omega \iff S(\tau \otimes \tau) = 0}$$

Let's see $S(\tau \otimes \tau) = 0$ more down-to-earth.

Pick a basis $\{v_1, v_2, \ldots, v_n\}$ of V $\rightsquigarrow$ $\Lambda^k V$ has a basis

$\{v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} \mid i_1 < i_2 < \ldots < i_k\}$

Basis

Explicitly:

$$Pl: \quad Gr(k, V) \longleftrightarrow \mathbb{P}(\Lambda^k V)$$

W – w/ basis $(w_1, \ldots, w_k) \longmapsto$ projection of $w_1 \wedge \ldots \wedge w_k$ ← in the basis $v_{i_1} \wedge \ldots \wedge v_{i_k}$

determines an $n \times k$ matrix A

all coeff-s are just $k \times k$-minors of matrix A.

$$\boxed{w_1 \wedge \ldots \wedge w_k = \sum_{I = \{i_1 < \ldots < i_k\}} v_{i_1} \wedge \ldots \wedge v_{i_k} \cdot P_I \quad (k \times k \text{ minor of } A).}$$

$= \det$ of

Then, Theorem 2 may be recast in a more familiar way:

For $\tau \in \Lambda^k V \setminus \{0\}$, we have:

$$\boxed{\begin{array}{l} \tau \in \Omega \iff \displaystyle\sum_{j\in J,\, j\notin I} (-1)^{\cdots} P_{I\cup\{j\}} P_{J\setminus\{j\}} = 0 \\ \text{for any } \cancel{\text{sets}}\ I, J \subseteq \{1, 2, \dots, n\} \text{ s.t. } |I| = k-1,\ |J| = k+1 \end{array}}$$

Exercise: Check this (in particular, determine power of (-1)).

Note: The eq-s in the RHS are known as Plücker relations and describe $Gr(k,V)$ as a projective variety

Proof of Thm 2

$\tau \in \Omega \iff S(\tau \otimes \tau) = 0.$

$\Rightarrow$ If $\tau = v_1 \wedge v_2 \wedge \dots \wedge v_k$ for some lin. ind $v_1, \dots, v_k \in V$

then complete this to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V.

$$\underbrace{S}_{\parallel}((v_1 \wedge \dots \wedge v_k) \otimes (v_1 \wedge \dots \wedge v_k)) \overset{?}{=} 0$$

$\sum_{j=1}^{n} \hat{v}_j \otimes \check{v}_j$, but for $j \le k$: $\hat{v}_j(\tau) = 0$

for $j > k$: $\check{v}_j(\tau) = 0$ follows!

✓

($\Leftarrow$) Know: $S(\tau \otimes \tau) = 0$ $\overset{?}{\Longrightarrow}$ τ - decomposable.

Consider the subspace $E \subseteq V$ via $E := \{v \in V \mid \hat{v}\tau = 0\}$

— // — $F \subseteq V^*$ via $F := \{f \in V^* \mid \check{f}\tau = 0\}$.

Easy: $E \subseteq F^{\perp} (\subset V)$ $\longleftarrow$ follows from $\hat{v}\check{f} + \check{f}\hat{v} = \mathrm{Id} \cdot f(v)$

Let: $r := \dim E$, $s := \dim F^{\perp}$, hence, $r \leq s$

Pick a basis $\{v_1, \ldots, v_n\}$ of V, so that $\{v_1, \ldots, v_r\}$ - basis of E

$\{v_1, \ldots, v_s\}$ - basis of $F^{\perp}$.

Note: For $i > s$, $v_i^* \in F$ by our definition. In particular, $\check{v}_i^*(\tau) = 0$ for $i > s$

Also: $\hat{v}_i(\tau) = 0$ for $i \leq r$ by def. of E.

Hence: $$S(\tau \otimes \tau) = \sum_{i=r+1}^{s} \underbrace{\hat{v}_i(\tau)} \otimes \check{v}_i^*(\tau)$$

$\boxed{\text{Claim}}$: $\{\hat{v}_i(\tau)\}_{i=r+1}^{s}$ - lin. indep. (otherwise you find lin. comb. of $\{v_i\}_{r+1 \leq i \leq s}$ in E $\Rightarrow$ $\Downarrow$)

$\Downarrow$

each $\check{v}_i^*(\tau) = 0$ $\xRightarrow[]{\text{for } i \leq s \;\; v_i^* \notin F}$ $\Downarrow$ unless the sum is empty, i.e. $r = s$.

<u>Conclusion</u>: $r = s \Rightarrow E = F^{\perp} \overset{?}{\Longrightarrow} \tau \in \Omega$.

<u>Actually</u>: τ is a multiple of $v_1 \wedge \ldots \wedge v_k = v_1 \wedge \ldots \wedge v_r$.

$$\left[\tau = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} c_{i_1, i_2, \ldots, i_k} \, v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} \right]$$

$$\hat{v}_i \tau = 0 \quad \forall \, i \leq r \overset{\text{Check!}}{\Longrightarrow} c_{i_1, \ldots, i_k} = 0 \text{ if } \{i_1, \ldots, i_k\} \not\supseteq \{1, \ldots, r\}.$$

$$\check{v}_i \tau = 0 \quad \forall \, i > r \overset{\text{Check!}}{\Longrightarrow} c_{i_1, \ldots, i_k} = 0 \text{ if } \{i_1, \ldots, i_k\} \not\subseteq \{1, \ldots, r\}.$$

$\Downarrow$ $\tau \neq 0 \Rightarrow k = r$ and furthermore:

$$\boxed{\tau = \text{multiple of } v_1 \wedge \ldots \wedge v_r}$$

$\Downarrow$

$\tau \in \Omega$