03/09/2021

Lecture #13

GL-orbit of a fixed one
$\parallel$
all decomposable wedges

Recall
$$Gr(k,V) = \Omega / \mathbb{C}^\times$$

Last time: $Gr(k,V)$, Plücker embedding, Plücker eq-s

$\Downarrow$

Today: ∞ - analogue

Def 1: a) Let $M(\infty) = Id + gl_\infty$, i.e. it consists of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ s.t. $a_{ij} - \delta_{ij}$ are all but fin. many ZERO.

b) $GL(\infty) \subseteq M(\infty)$ - a subset of invertible matrices.

$\boxed{\text{Exercise (Hwk)}}$: (a) $GL(\infty)$ - group, $M(\infty)$ - monoid (under usual multiplication)

(b) $M(\infty) \curvearrowright \mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2}, m} V$ & group action $GL(\infty) \curvearrowright \mathcal{F}^{(m)}$

Note: $A(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) \overset{def}{=} (Av_{i_0}) \wedge (Av_{i_1}) \wedge (Av_{i_2}) \wedge \ldots$

Def 2: Define $\Omega \subseteq \mathcal{F}^{(0)}$ as $\boxed{\Omega = GL(\infty)\psi_0}$

This is a natural analogue of one definition from last time in fin. dim. case.
The next result shows an alternative description from last time applies to some extent

Lemma 1: For any $i_0 > i_1 > i_2 > \dots$ s.t. $i_k = -k$ $\forall k \gg 0$, we have

$$\boxed{v_{i_0} \wedge v_{i_1} \wedge \dots \in \Omega}$$

▶

$$\psi_0 = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots \wedge v_{-N} \wedge v_{-N-1} \wedge \dots$$

$$\psi' = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{-N} \wedge v_{-N-1} \wedge \dots$$ as $i_k = -k$ for k big enough.

There exists a permutation $\sigma: \mathbb{Z} \to \mathbb{Z}$, $\sigma(k) = k$ for $|k| \gg 0$

s.t. $\sigma(k) = i_k$ $\forall k \geq 0$.

$\rightsquigarrow$ defines an elt $A \in GL(\infty)$ s.t. $A\psi_0 = \psi'$ ∎

Def 3: For any $m \in \mathbb{Z}$, define a linear operator

$$\boxed{\begin{array}{l} S: \mathcal{F}^{(m)} \otimes \mathcal{F}^{(m)} \longrightarrow \mathcal{F}^{(m+1)} \otimes \mathcal{F}^{(m-1)} \\ \boxed{S := \sum_{i \in \mathbb{Z}} \hat{v}_i \otimes \check{v}_i} \end{array}}$$

$\leftarrow$ Exactly the analogue of S from last time (in fin. dim. setup)

recall: $\mathcal{F}^{(m)} = \Lambda^{\frac{\infty}{2}, m} V$

Note: For any $u \otimes v \in \mathcal{F}^{(m)} \otimes \mathcal{F}^{(m)}$, only fin. many of $\hat{v}_i(u) \otimes \check{v}_i(v) \neq 0$.

(as $\hat{v}_i(u) = 0$ for $i \ll 0$ while $\check{v}_i(v) = 0$ for $i \gg 0$)

$\boxed{\text{Thm 1}}$: For $\tau \in \mathcal{F}^{(0)} \setminus \{0\}$, we have

$$\boxed{\underbrace{\tau \in \Omega}_{GL(\infty)\psi_0} \iff \underbrace{S(\tau \otimes \tau) = 0}_{S:\ \mathcal{F}^{(0)} \otimes \mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(1)} \otimes \mathcal{F}^{(-1)}}}$$

Exercise (Hwk): Prove Thm

Hint: Reduce to the fin. dim. case, where we proved similar Theorem last time.

Def: The semiinfinite Grassmannian Gr is defined by $\boxed{Gr = \Omega / \mathbb{C}^\times}$

Identifying $v_i \in V$ ($i \in \mathbb{Z}$) with $t^{-i} \in \mathbb{C}((t))$, we can interpret Gr as follows:

$$! \quad Gr = \left\{ \underset{\text{subspace}}{E \subseteq \mathbb{C}((t))} \;\middle|\; \begin{array}{l} t^k \mathbb{C}[[t]] \subseteq E \text{ for } k \gg 0 \\ \dim (E / t^k \mathbb{C}[[t]]) = k \text{ for these } k \end{array} \right\}$$

work through at home

Rmk: (a) $t^k \mathbb{C}[[t]] \subseteq E$ & $\dim(E/t^k\mathbb{C}[[t]]) = k \Rightarrow$ same holds $\forall k' > k$.

(b) $\forall E \in Gr\ \exists k \gg 0$ s.t. $E \subseteq t^{-k}\mathbb{C}[[t]]$

$$\rightsquigarrow t^k \mathbb{C}[[t]] \subseteq E \subseteq t^{-k} \mathbb{C}[[t]] \rightsquigarrow \underbrace{E/t^k\mathbb{C}[[t]] \subset t^{-k}\mathbb{C}[[t]]/t^k\mathbb{C}[[t]]}_{!\ \text{defines a point in } Gr(k, 2k).}$$

Corollary : $\boxed{Gr = \underbrace{\bigcup}_{\text{nested union of finite Grassmannians}} Gr(k, 2k)}$

$$\begin{array}{ccccl} \sigma: & \mathcal{F} & \xrightarrow{\sim} & \mathcal{B} & \\ & \cup & & \cup & \\ & \mathcal{F}^{(0)} & \xrightarrow{\sim} & \mathcal{B}^{(0)} & = \mathbb{C}[x_1, x_2, x_3, \dots]. \\ & \cup & & & \\ & \tau & & & \end{array}$$

$\boxed{\text{Key Goal for today}}$: Rewrite Plücker Eq-n $S(\tau \otimes \tau) = 0$ on the "bosonic side", study application to ODE

Recall: quantum fields

$$\boxed{\begin{aligned} X(u) &= \sum_{i\in\mathbb{Z}} \hat{v}_i \cdot u^i \\ X^*(u) &= \sum_{i\in\mathbb{Z}} \check{v}_i\, u^{-i} \end{aligned}}$$

← wedging operators

← contracting operators

$$\underbrace{S}_{\sum_{i\in\mathbb{Z}} \hat{v}_i \otimes \check{v}_i}(\tau\otimes\tau)=0 \overset{\text{obvious}}{\iff} \boxed{\underbrace{CT_u}_{\text{coefficient of } u^0}\left(X(u)\tau \otimes X^*(u)\tau\right) = 0}$$

← Fermionic side

$$\mathcal{F}^{(0)} \simeq \mathcal{B}^{(0)}$$

$$\tau \in \mathcal{F}^{(0)} \longleftrightarrow \tau \in \mathcal{B}^{(0)}$$

$$\Downarrow$$

$$\boxed{CT_u\left(\Gamma(u)\tau \otimes \Gamma^*(u)\tau\right) = 0}$$

← Bosonic side

$$\tau \in \mathcal{B}^{(0)} = \mathbb{C}[x_1, x_2, x_3, \ldots]$$

$$\Downarrow$$

$$\tau\otimes\tau \in \mathbb{C}[x_1', x_1'', x_2', x_2'', x_3', x_3'', \ldots]$$

Recalling the explicit f-las for $\Gamma(u)$, $\Gamma^*(u)$ we get:

↓

$$CT_u\left(u\cdot e^{\sum_{j>0} x_j' u^j}\cdot e^{-\sum_{j>0}\frac{1}{j}\partial/\partial x_j' u^{-j}}\cdot e^{-\sum_{j>0} x_j'' u^j}\, e^{\sum_{j>0}\frac{1}{j}\partial/\partial x_j'' u^{-j}}\,\tau(x_1',x_2',\ldots)\,\tau(x_1'',x_2'',\ldots)\right)=0$$

$$\Updownarrow$$

$$\boxed{CT_u\left(u\cdot \exp\Big(\sum_{j>0}(x_j'-x_j'')u^j\Big)\cdot \exp\Big(\sum_{j>0}\Big(\frac{\partial}{\partial x_j'}-\frac{\partial}{\partial x_j''}\Big)\frac{u^{-j}}{-j}\Big)\right)\tau(x')\tau(x'')=0}$$

<u>Corollary</u>: $\tau\in\mathcal{B}^{(0)}$ satisfies $\sigma^{-1}(\tau)\in\Omega$ if and only if above eq-n holds.

$\boxed{\text{Change of variables}}$: $\begin{cases} x'=x-y \\ x''=x+y\end{cases}$, i.e. $\begin{matrix} x_i'=x_i-y_i \\ x_i''=x_i+y_i\end{matrix}$ $\forall i$ $\qquad \mathbb{C}[x_1',x_1'',x_2',x_2'',\ldots]=\mathbb{C}[x_1,y_1,x_2,y_2\ldots]$

$$\Downarrow$$

$$\begin{cases} x'-x''=-2y \\ \dfrac{\partial}{\partial x_i'}-\dfrac{\partial}{\partial x_i''}=-\dfrac{\partial}{\partial y_i}\end{cases}$$

$$\boxed{CT_u\left(u\cdot \exp\Big(-2\sum_{j>0}u^j y_j\Big)\cdot \exp\Big(\sum_{j>0}\frac{u^{-j}}{j}\cdot\frac{\partial}{\partial y_j}\Big)\,\tau(x-y)\,\tau(x+y)\right)=0}$$

Def: Given any $P(x) = \mathbb{C}[[x_1, x_2, \dots]]$ (series), $f(x), g(x) \in \mathbb{C}[x_1, x_2, \dots]$ (polynomials) define $A(P, f, g) \in \mathbb{C}[x_1, x_2, \dots]$ via

$$\boxed{A(P,f,g) := \Big[P\big(\underbrace{\tfrac{\partial}{\partial z}}_{\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots\right)}\big) \big(f(x-z)\cdot g(x+z)\big) \Big]_{|z=0}}$$

Rmk: (a) If $P_-(x) := P(-x) \Rightarrow A(P, f, g) = A(P_-, g, f)$

$\Downarrow$

(b) If $\boxed{P \text{ is odd } (P_-(x) = -P(x)) \Rightarrow A(P, f, f) = 0}$ $\longleftarrow$ We shall use this in what follows!

Thm 2 (Hirota bilinear relations): For $\tau \in B^{(0)} \setminus \{0\}$, we have $\sigma^{-1}(\tau) \in \Omega$ iff

$$\boxed{A\Big(\sum_{j\geq 0} S_j(-2y) S_{j+1}(\tilde{x}) \exp\big(\sum_{s>0} y_s x_s\big), \tau, \tau\Big) = 0 \qquad \text{where } \tilde{x}_1 = x_1,\ \tilde{x}_2 = \frac{x_2}{2},\ \tilde{x}_3 = \frac{x_3}{3}, \dots} \quad (*)$$

! NOTE: View as a series in x's with coeff-s in independent y-variables

Recall: (from the very def-n)

$$\sum_{j\geq 0} S_j(-2y)u^j = \exp\Big(-\sum_{j\geq 1} 2y_j u^j\Big)$$

$$\sum_{j\geq 0} S_j(\tilde{\partial}_y)u^{-j} = \exp\Big(\sum_{j\geq 1} \frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\Big)$$

$$\tilde{\partial}_{y_1} = \partial/\partial y_1,\quad \tilde{\partial}_{y_2} = \frac{1}{2}\partial/\partial y_2,\ \ldots$$

Proof of Thm 2

$$0 = CT_u\Big(u\cdot \exp\Big(-\sum_{j\geq 1} 2y_j u^j\Big)\cdot \exp\Big(\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\Big)\tau(x+y)\tau(x-y)\Big)$$

! Introduce yet another variables

$t = (t_1, t_2, t_3, \ldots)$

$$CT_u\Big(u\cdot \exp(\ldots)\exp(\ldots)\ \tau(x+y+t)\tau(x-y-t)\Big)\Big|_{t=0}$$

$\overline{\overline{\text{above f-las}}}$

$$CT_u\Big(u\cdot\Big(\sum_{j\geq 0} S_j(-2y)u^j\Big)\Big(\sum_{j\geq 0} S_j(\tilde{\partial}_y)u^{-j}\Big)\cdot \tau(x+y+t)\tau(x-y-t)\Big)\Big|_{t=0} =$$

As all dependence on y,t is actually on $y+t$

replace $\tilde{\partial}_y$ by $\tilde{\partial}_t$

$$= CT_u\Big(u\cdot\Big(\sum_{j\geq 0} S_j(-2y)u^j\Big)\Big(\sum_{j\geq 0} S_j(\tilde{\partial}_t)u^{-j}\Big)\tau(x+y+t)\tau(x-y-t)\Big)\Big|_{t=0}$$

By def-n of CT_u

$$= \Big[\Big(\sum_{j\geq 0} S_j(-2y)S_{j+1}(\tilde{\partial}_t)\Big)\underbrace{\tau(x+y+t)\tau(x-y-t)}_{\overset{\text{Taylor decomposit.}}{=}\exp\left(\sum_{s\geq 1} y_s\frac{\partial}{\partial t_s}\right)\tau(x+t)\tau(x-t)}\Big]\Big|_{t=0}$$

$$\|$$

$$\underbrace{\left[\left(\sum_{j\geq 0} S_j(-2y)\, S_{j+1}(\tilde{\partial}_t)\exp\left(\sum_{s\geq 1} y_s \frac{\partial}{\partial t_s}\right)\right)\tau(x+t)\tau(x-t)\right]\Big|_{t=0}}$$

THUS: $\boxed{A\left(\sum_{j\geq 0} S_j(-2y)\, S_{j+1}(\tilde{x})\exp\left(\sum_s y_s x_s\right),\ \tau,\ \tau\right) = 0}$

This is actually a family of equations!

i.e. each monomial $y_1^{\gamma_1} y_2^{\gamma_2} \cdots$ produces its own equation

Let's look what the simplest ones are.

- $\gamma_1 = \gamma_2 = \dots = 0$ (i.e. no y's)

Get: $A(\underbrace{S_1(\tilde{x})}_{\substack{\parallel \\ x_1}}, \tau, \tau) = 0$ ← vacuous!

x_1 ← odd polynomial

(as we know $A(P, f, f) = 0$ for odd P)

• $\gamma_r = 1$ & $\gamma_s = 0$ $\forall s \neq r$

The coeff. of monomial y_r in $\sum_{j\geq 0} S_j(-2y)\, S_{j+1}(\tilde{x}) \exp(\sum y_s x_s)$

is: $\underbrace{x_1 x_r - 2 S_{r+1}(\tilde{x})}$

denote by $T_r(x)$.

* $\underline{r=1}$: $T_1(x) = x_1^2 - 2 \underbrace{S_2(\tilde{x})}_{\frac{x_1^2}{2} + x_2} = -2x_2$ $\Longrightarrow$ don't get any nontrivial eq-n.

↑ odd in x-variables

* $\underline{r=2}$: $T_2(x) \overset{\text{Exercise}}{=} -\frac{x_1^3}{3} - \frac{2x_3}{3}$ ⟵ odd in x's ⟹ nothing interesting eq-n.

! $\underline{\underline{r=3}}$: $\underbrace{T_3(x) \overset{\text{Exercise}}{=} \frac{x_1 x_3}{3} - \frac{x_4}{2} - \frac{x_2^2}{4} - \frac{x_1^4}{12} - \frac{x_1^2 x_2}{2}}$ ⟵ This provides the 1st nontrivial eq-n!

$\Downarrow$

$\underline{\underline{\text{eq-n}}}$: $\boxed{A(T_3(x), \tau, \tau) = 0}$

$\Downarrow$ ignoring odd terms

$$\boxed{A\left(\frac{x_1 x_3}{3} - \frac{x_2^2}{4} - \frac{x_1^4}{12}, \tau, \tau\right) = 0}$$

$$\Downarrow$$

$$\left[\left(\frac{1}{3}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3}-\left(\frac{\partial}{\partial z_2}\right)^2\cdot\frac{1}{4}-\left(\frac{\partial}{\partial z_1}\right)^4\cdot\frac{1}{12}\right)\tau(x-z)\tau(x+z)\right]_{|z=0}=0.$$

$$\Downarrow$$

$$(**)\quad \boxed{\left[\left(\partial_{z_1}^4+3\partial_{z_2}^2-4\partial_{z_1}\partial_{z_3}\right)\tau(x-z)\tau(x+z)\right]_{|z=0}=0}$$

$x=(x_1,x_2,x_3,x_4,\_\_)$

$z=(z_1,z_2,z_3,\_\_)$

Fix $x_m=c_m$ for $m>3$, rename $x_1=x$, $x_2=y$, $x_3=t$

Set: $\boxed{u:=2\partial_x^2\log\tau}$ $\leftarrow$ $u=u(x,y,t)$

Prop (Hwk Exercise*): τ satisfies $(**)$ iff u satisfies KP eq-n:

$$\boxed{\frac{3}{4}\partial_y^2u=\partial_x\left(\partial_t u-\frac{3}{2}u\cdot\partial_x u-\frac{1}{4}\partial_x^3u\right)}$$

The take home from today's class is:

$\boxed{\text{Corollary:}}$ Any element of Gr provides a solution of KP eq-n.

- $\tau\in\Omega \rightsquigarrow u$ - sol-n

  $Gr=\Omega/\mathbb{C}^\times \rightsquigarrow u$ - same in $\mathbb{C}^\times$-orbit

$\underline{\underline{\text{Last week}}}$: $\underbrace{\mathcal{F}^{(0)} \ni v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots}_{\substack{\cap \ (\text{Lemma 1}) \\ \Omega}} \xrightarrow{\sigma} \underset{\substack{\cap \\ \mathcal{B}^{(0)}}}{S_\lambda}$ - Schur pol-s.

$$\Downarrow$$

$\boxed{\text{Corollary}}$: For any partition λ and any collection of constants $c_4, c_5, \ldots$, get the following solution of the KP eq-n:

$$\boxed{u = 2\partial_x^2 \log S_\lambda(x, y, t, c_4, c_5, \ldots)}$$

Recall from Lecture 11 (used to define $gl_\infty \curvearrowright B^{(0)}$)

$$\Gamma(u,v) = \exp\Big(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j}\Big) \exp\Big(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j\Big)$$

$$\| $$

$$:\Gamma(u)\Gamma^*(v):$$

$\boxed{\text{Thm 3}}$: If $\tau \in \Omega \Rightarrow (1 + a\cdot\Gamma(u,v))\tau \in \underbrace{\Omega_{u,v}}_{\{\tau \in B^{(0)}((u,v)) \mid S(\tau\otimes\tau)=0\}}$ $\forall a \in \mathbb{C}$.

$\Downarrow$ $\psi_0 \longleftrightarrow \mathbb{1} \in B^{(0)}$

$\boxed{\text{Cor}}$: $\forall a_1, \dots, a_n \in \mathbb{C}$ we get

$$\tau = (1 + a_1\Gamma(u_1,v_1))(1 + a_2\Gamma(u_2,v_2)) \dots (1 + a_n\Gamma(u_n,v_n))\mathbb{1} \in \Omega_{u_1,v_1,\dots,u_n,v_n}$$

$$\Downarrow$$

$\boxed{\text{Cor}}$: $\boxed{u = 2\left(\frac{\partial}{\partial x_1}\right)^2 \log\tau \text{ is a solution of the KP eq-n}}$

it is known as the "n-soliton" solution

$\boxed{\text{Example}}$: $n=1$, $\tau = (1 + a\,\Gamma(u,v))\mathbb{1} = 1 + a\cdot e^{(u-v)x + (u^2-v^2)y + (u^3-v^3)t + \underbrace{c}}$

constant, which can be further absorbed into a!

$$\Downarrow$$

$$u(x,y,t) = 2\partial_x^2 \log\tau = \frac{(u-v)^2}{2}\cdot\frac{1}{\cosh^2 \frac{1}{2}((u-v)x + (u^2-v^2)y + (u^3-v^3)t)}.$$

$\Downarrow$ set $v=-u$ to eliminate y

$$\boxed{u(x,t) = \frac{2u^2}{\cosh^2(ux + u^3 t)}}$$

By above: it is a solution of KdV equation.

We shall prove Thm 3 either next time or will postpone it to hwk.