Lecture # 14

03/11/2021

Last time:

- $\Omega = GL(\infty)\psi_0 \subseteq \mathcal{F}^{(0)} \longleftrightarrow \Omega \subseteq \mathcal{B}^{(0)} = \mathbb{C}[x_1, x_2, \dots]$

  contains all Schur polyn.

  Thm: $\tau \in \mathcal{F}^{(0)} \setminus \{0\}$ is in $\Omega \iff \boxed{S(\tau \otimes \tau) = 0}$

  $S = \sum_{i \in \mathbb{Z}} \hat{v}_i \otimes \check{v}_i : \mathcal{F}^{(0)} \otimes \mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(1)} \otimes \mathcal{F}^{(-1)}$.

  ([Kac – Raina, Prop 7.2])

- Thm: $\tau \in \mathcal{B}^{(0)} \setminus \{0\}$ is in Ω iff:

$$\boxed{CT_u\left(u \cdot \exp\left(-2\sum_{j>0} u^j y_j\right) \exp\left(\sum_{j>0} \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}\right) \tau(x-y)\tau(x+y)\right) = 0}$$

$$\Updownarrow$$

- $A(P, f, g) := \left[P\left(\frac{\partial}{\partial z}\right)\left(f(x-z)g(x+z)\right)\right]_{|z=0}$

[Kac–Raina Thm 7.1]

$$\boxed{A\left(\sum_{j\geq 0} S_j(-2y) S_{j+1}(\tilde{x}) \exp\left(\sum_{s\geq 1} y_s x_s\right), \tau, \tau\right) = 0}$$

! It is a family of eq-s, hence the name, KP hierarchy.

Looking at coeff. of y_3^1, get:

$$\boxed{\left.\left(\left(\partial_{z_1}^4+3\partial_{z_2}^2-4\partial_{z_1}\partial_{z_3}\right)\tau(x-z)\tau(x+z)\right)\right|_{z=0}=0}$$

$x=x_1$

$y=x_2$

$t=x_3$

$x_{>3}$ – set to constants $c_4, c_5, \ldots$

$\Updownarrow$ Hwk 7

$\boxed{u:=\alpha\partial_x^2\log\tau(x,y,t,c_4,c_5,\ldots)}$ to be a solution of

KP eq-n: $$\boxed{\frac{3}{4}\partial_y^2 u=\partial_x\Big(\underbrace{\partial_t u-\frac{3}{2}u\partial_x u-\frac{1}{4}\partial_x^3 u}_{\text{KdV eqn}}\Big)}$$

! **Take Home**: Every point of $\underbrace{\Omega/\mathbb{C}^\times}_{Gr}$ provides a solution of the KP eq-n (& hierarchy).

semi-infinite Grassmannian

Last time we concluded with the following:

$$\boxed{\underbrace{\Gamma(u,v)}_{\text{determines glob } \curvearrowright \mathcal{B}^{(0)}} =: \Gamma(u)\Gamma^*(v) := \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_j\right) \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j\right)}$$

If $\tau \in \Omega \subseteq \mathcal{B}^{(0)}$

$\Downarrow$

$\boxed{\text{Thm 3 (Tue)}}$: $\big(1 + \underset{\mathbb{C}}{\overset{}{a}}\,\Gamma(u,v)\big)\tau \in \Omega_{u,v} = \{\tau' \in \mathcal{B}^{0}((u,v)) \mid S(\tau' \otimes \tau') = 0\}$.

Application: $\underset{\mathcal{F}^{(0)}}{\psi_0} \Longleftrightarrow \mathbb{1} \in \mathcal{B}^{(0)} \rightsquigarrow$ get a family

$$\boxed{\tau_{N;\underline{a},\underline{u},\underline{v}} = (1 + a_1\Gamma(u_1,v_1))(1 + a_2\Gamma(u_2,v_2)) \ldots (1 + a_n\Gamma(u_n,v_n))\mathbb{1} \in \Omega_{\underline{u},\underline{v}}}$$

i.e. satisfy $S(- \otimes -) = 0$.

Remarks: Prove Thm 3. — today!

Starting point :

$$\boxed{\begin{aligned}
\Gamma(u)\Gamma(v) &= \frac{u-v}{u} \quad :\Gamma(u)\Gamma(v): \\
\Gamma(u)\Gamma^*(v) &= \frac{u}{u-v} \quad :\Gamma(u)\Gamma^*(v): \\
\Gamma^*(u)\Gamma(v) &= \frac{u}{u-v} \quad :\Gamma^*(u)\Gamma(v): \\
\Gamma^*(u)\Gamma^*(v) &= \frac{u-v}{u} \quad :\Gamma^*(u)\Gamma^*(v):
\end{aligned}}$$

▶ $\Gamma(u)_{|\mathcal{B}^{(m)}} = u^{m+1} \cdot z \cdot \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)$

$\Gamma(u)\Gamma(v) \rightsquigarrow :\Gamma(u)\Gamma(v): \times$ extra term coming from reordering :

$$\underbrace{\exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \exp\left(\sum \frac{a_{-j}}{j} v^j\right)}_{\text{taking } a_{>0} \text{ to the right of } a<0}$$

$$\Rightarrow \text{factor} = \exp\left(-\sum_{j>0} \frac{j}{j \cdot j} \cdot \frac{v^j}{u^j}\right) = \exp\left(-\underbrace{\sum_{j>0} \frac{1}{j}\left(\frac{v}{u}\right)^j}_{\log(1-\frac{v}{u})}\right) = \frac{u-v}{u}$$

This is similar to Lecture 11 & Hwks.

! <u>Note</u> : In case of $\frac{u}{u-v}$, it should be actually perceived as a series in the region $|u|>|v|$

i.e. $\frac{u}{u-v} = \frac{1}{1-\frac{v}{u}} = 1 + \frac{v}{u} + \frac{v^2}{u^2} + \frac{v^3}{u^3} + \ldots$

Let's use $\Gamma_+(u)$, $\Gamma_-(u)$ instead of $\Gamma(u)$, $\Gamma^*(u)$, respectively.

$\boxed{\text{Cor 1}}$ $\Gamma_{\varepsilon_1}(u_1)\ldots\Gamma_{\varepsilon_n}(u_n) = :\Gamma_{\varepsilon_1}(u_1)\cdot\ldots\cdot\Gamma_{\varepsilon_n}(u_n): \times \underbrace{\prod_{1\le i<j\le n}\left(\frac{u_i-u_j}{u_i}\right)^{\varepsilon_i\varepsilon_j}}_{\text{rat. f-n expanded as a series in the region } |u_1|>|u_2|>\ldots>|u_n|.}$

$\varepsilon_i \in \{\pm\} = \{\pm 1\}$

$\Downarrow$

$\boxed{\text{Cor 2}}$ All matrix coeff-s of $\Gamma_{\varepsilon_1}(u_1)\ldots\Gamma_{\varepsilon_n}(u_n)$ are series in $u_1,\ldots,u_n$ which converge to rat-l f-s of the form $\prod_{i<j}\left(1-\frac{u_j}{u_i}\right)^{\varepsilon_i\varepsilon_j}\times \underbrace{P}_{\in\mathbb{C}[u_1^{\pm1},\ldots,u_n^{\pm1}]}$.

Rmk: In particular, we can substitute formal variables u,v with complex numbers unless those rational f-s have poles at them

$\boxed{\text{Cor 3}}$ $\Gamma(u',v')\,\Gamma(u,v) = :\Gamma(u',v')\,\Gamma(u,v):\ \cdot \dfrac{(u'-u)(v'-v)}{(u'-v)(v'-u)}$

▶ $\Gamma(u',v') \overset{\text{def}}{=} :\Gamma_+(u')\,\Gamma_-(v'): \longleftarrow$ recall

$$\left.\begin{aligned} \Gamma_+(u')\,\Gamma_-(v') &= \overbrace{:-\!\!\shortparallel\!\!-:}^{=\Gamma(u',v')} \cdot \frac{u'}{u'-v'} \quad (\text{Cor } 1) \\ \Gamma_+(u)\,\Gamma_-(v) &= \underbrace{:-\!\!\shortparallel\!\!-:}_{=\Gamma(u,v)} \cdot \frac{u}{u-v} \quad (-\!\shortparallel\!-) \end{aligned}\right\} \Rightarrow \Gamma(u',v')\,\Gamma(u,v) = \underbrace{\frac{u'-v'}{u'} \cdot \frac{u-v}{u}}\ \times \underbrace{\Gamma_+(u')\,\Gamma_-(v')\,\Gamma_+(u)\,\Gamma_-(v)}$$

$\parallel$

By Cor 1: $\underbrace{\dfrac{u'}{u'-v'} \cdot \dfrac{u'-u}{u'} \cdot \dfrac{u'}{u'-v} \cdot \dfrac{v'}{v'-u} \cdot \dfrac{v'-v}{v'} \cdot \dfrac{u}{u-v}} \cdot :\Gamma_+(u')\,\Gamma_-(v')\,\Gamma_+(u)\,\Gamma_-(v):$

$\Downarrow$

$$\boxed{\Gamma(u',v')\,\Gamma(u,v) = \frac{(u'-u)(v'-v)}{(u'-v)(v'-u)} :\Gamma(u',v')\,\Gamma(u,v):}$$

■

$\boxed{\text{Cor 4}}$ If $u \neq v$: $\boxed{\lim_{\substack{u'\to u \\ v'\to v}} \Gamma(u',v')\,\Gamma(u,v) = 0}$

See Rmk on p. 5

$\Updownarrow$

"$\Gamma(u,v)^2 = 0$" in Kac-Raina

⑥

Proof of Thm 3 (from Tue)

$$\begin{matrix} \tau \in \Omega \\ S(\tau\otimes\tau)=0 \end{matrix} \overset{?}{\Longrightarrow} \begin{matrix} (1+a\,\Gamma(u,v))\tau \in \Omega_{u,v} \\ S(-\otimes-)=0. \end{matrix}$$

$$\underbrace{S\big((1+a\Gamma(u,v))\tau \otimes (1+a\Gamma(u,v))\tau\big)}_{\|} \overset{?}{=} 0$$

$$\underbrace{S(\tau\otimes\tau)}_{=0} + a\cdot \underbrace{S\big(\Gamma(u,v)\tau\otimes\tau + \tau\otimes\Gamma(u,v)\tau\big)}_{\overset{(*)}{=}0}$$

$$+ a^2 S\big(\Gamma(u,v)\tau\otimes\Gamma(u,v)\tau\big) \longleftarrow \text{Why is this guy ZERO?}$$

$$S=\sum_{i\in\mathbb{Z}} \hat{V}_i\otimes\check{V}_i \;!\; \xRightarrow{\text{Easy Exercise}} \boxed{S(x\otimes 1+1\otimes x)=(x\otimes 1+1\otimes x)S} \quad (*)$$

$$S(\Gamma(u,v)\tau\otimes\Gamma(u,v)\tau)=\lim_{\substack{u'\to u\\ v'\to v}} \frac{1}{2} S\big(\Gamma(u,v)\tau\otimes\Gamma(u',v')\tau + \Gamma(u',v')\tau\otimes\Gamma(u,v)\tau\big)$$

$$= \lim_{\substack{u'\to u\\ v'\to v}} \frac{1}{2} S\Big(\big(\Gamma(u,v)\otimes 1 + 1\otimes\Gamma(u,v)\big)\big(\Gamma(u',v')\otimes 1+1\otimes\Gamma(u',v')\big)(\tau\otimes\tau)\Big)$$

$$- \lim_{\substack{u'\to u\\ v'\to v}} \frac{1}{2} S\Big(\underbrace{\Gamma(u,v)\Gamma(u',v')\tau\otimes\tau}_{\lim_{u'\to u, v'\to v}\to 0 \text{ (by Cor 4)}} + \tau\otimes\underbrace{\Gamma(u,v)\Gamma(u',v')\tau}_{\lim_{u'\to u, v'\to v}\to 0 \text{ (by Cor 4)}}\Big)$$

$$=0-0=0$$

For the rest of today — irreducibility of Virasoro Verma modules ([Kac-Raina] §8)

$\mathbb{C}^2 \ni \lambda = (c, h) \rightsquigarrow M_\lambda = M^+_{c,h}$ — Verma module / Vir

c-weight L_0-weight

From first 3 weeks: $M_{c,h}$ - irred. for generic $(c,h) \in \mathbb{C}^2$.

Key Tool: $M^+_\lambda \times M^-_{-\lambda} \xrightarrow[\text{invariant pairing}]{(\cdot,\cdot)_\lambda} \mathbb{C}$ $\xrightarrow{\text{Lectures 5-6}}$ $M^+_\lambda \times M^+_\lambda \xrightarrow{(\cdot,\cdot)} \mathbb{C}$ ← contravariant form

"Shapovalov form"

Determined by: $(1, 1) = 1$

$(L_n v, w) = (v, L_{-n} w)$. $\forall n$, $\forall v, w \in M^+_\lambda$

Recall: M_λ - irred $\Longleftrightarrow$ Shapovalov form is non-degenerate!

$$M_\lambda^+ \underset{\mathbb{Z}\text{-gr.v.sp.}}{\simeq} \mathcal{U}(\underbrace{Vir^-}_{\{L_n\}_{n<0}}) \qquad Vir^- = \bigoplus_{n<0} \mathbb{C}L_n$$

$$\rightsquigarrow$$

degree n part of determ

$$\boxed{\det_n(c,h) = \det\left((X_I v_\lambda, X_J v_\lambda)_{I,J} \right)}$$

$\{X_I\}$ - a basis of $\mathcal{U}(Vir^-)[-n]$

$$\boxed{\underset{(c,h)}{\underset{\shortparallel}{M_\lambda}} \text{ - irred} \iff \det_n(c,h) \neq 0 \quad \forall n}$$

Note: "$\det_n \neq 0$" does not depend on a choice of a basis!

Recall: $\det_n(c,h) = 0$ for some $n \iff \exists$ singular vector in M_λ of degree $\geq -n$.

$\Downarrow$

$\boxed{\text{Cor:}}$ $\det_n(c,h) = 0 \Rightarrow \det_{n+k}(c,h) = 0 \quad \forall k \geq 0.$

Examples (from Lecture 5):

$$\left.\begin{aligned} \det_1(c,h) &= -2h \\ \det_2(c,h) &= -4h\left((2h+1)(4h+\tfrac{c}{2}) - 9h\right) \end{aligned}\right\} \text{ in particular } \det_1 \text{ divides } \det_2$$

Theorem 1 (Feigin – Fuchs, Kac)

$$\det_n(c,h) = \underbrace{K_n}_{\substack{\text{nonzero constant} \\ \text{(see next page)}}} \times \prod_{\substack{r,s\geq 1 \\ rs \leq n}} \big(h - h_{r,s}(c)\big)^{\underbrace{p}_{\text{partition f-n}}(n-rs)}$$

$$h_{r,s}(c) := \frac{1}{48}\Big((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}\,(r^2-s^2) - 24rs - 2 + 2c \Big)$$

choose the same branch of $\sqrt{\cdots}$ for all r,s

$\boxed{\text{Thm 2.}}$ The leading term in h in $\det_n(c,h)$ equals

$$\underbrace{K_n}\cdot h^{\sum_{r,s\geq 1}^{rs\leq n} p(n-rs)}$$

$$K_n := \prod_{\substack{r,s\geq 1\\ rs\leq n}} \left((2r)^s\cdot s!\right)^{\overbrace{p(n-rs)-p(n-r(s+1))}^{=: m(r,s)}}$$

the f-la for K_n is on the next hwk

At the deg $-n$, the basis of $\mathcal{U}(Vir)$ consists of:

$$\boxed{\{L_{-n}^{k_n}\cdots L_{-2}^{k_2}L_{-1}^{k_1} \mid k_1+2k_2+\ldots+nk_n=n\}}$$

We are computing $\det_n = \det\left(\left(L_{-n}^{k_n}\cdots L_{-1}^{k_1}v_\lambda,\ L_{-n}^{k'_n}\cdots L_{-1}^{k'_1}v_\lambda\right)\right)$

$\boxed{\text{Claim 1.}}$ The leading power of h is coming from diagonal terms.

$$\left(L_{-n}^{k_n}\cdots L_{-1}^{k_1}v_\lambda,\ L_{-n}^{k_n}\cdots L_{-1}^{k_1}v_\lambda\right) = \left(v_\lambda,\ L_1^{k_1}\cdots L_n^{k_n}L_{-n}^{k_n}\cdots L_{-1}^{k_1}v_\lambda\right)$$

$\boxed{\text{Claim 2}}$: The leading power of h on such diagonal terms is

$$k_1+k_2+\ldots+k_n.$$

You'll need to work out both Claims while proving f-la for K_n in the homework

Remarks : $\displaystyle\sum_{\mu\vdash n \text{ (partition of } n)} \sum_i \underbrace{k_i(\mu)}_{\text{number of } i^{th} \text{ in } \mu} \overset{?}{=\!=} \sum_{\substack{r,s\geq 1\\ rs\leq n}} p(n-rs)$

Recall : $m(r,s) = p(n-rs) - p(n-r(s+1)) = \#\{\mu\vdash n \mid r \text{ occurs in } \mu \text{ exactly } s \text{ times}\}$

$$\Downarrow$$

$$\sum_{\mu\vdash n}\sum_i k_i(\mu) = \sum_{\substack{r,s\\ rs\leq n}} s\cdot m(r,s) = \sum_{\substack{r,s\\ rs\leq n}} s\,(p(n-rs)-p(n-r(s+1)))$$

$$= \sum_r\Big(\sum_s \underbrace{(s-(s-1))}_{1}\, p(n-rs)\Big) = \sum_{r,s} p(n-rs)$$

Thm 3 For $r,s \geq 1$:

$$\boxed{\det_{rs}(c, h_{r,s}(c)) = 0}$$

To be proved later on

To prove Thm 1, it remains to show

$\det_n(c,h)$ has a degree $p(n-rs)$ zero at $h = h_{r,s}(c)$

Lemma: Let $A(t)$ be a matrix with entries polynomial in t, s.t. $\dim(\operatorname{Ker} A(0)) \geq n \Rightarrow \det A(t) \vdots t^n$.

▷ Pick a basis of V s.t. $v_1, \ldots, v_n$ - basis for $\operatorname{Ker} A(0)$

$A(t) : V \to V$

$\Downarrow$

the first n columns of $A(t)$ are divisible by t

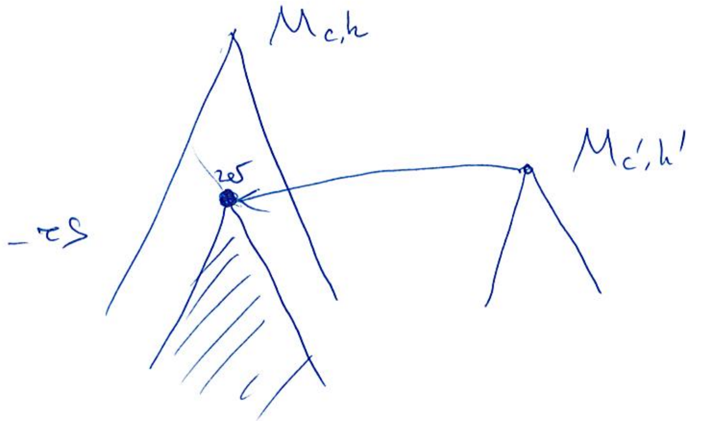
$\Downarrow$

$$\boxed{\det A(t) \vdots t^n}$$

■

Proof of Theorem 1

- $\det_{rs}(c, h_{r,s}(c)) = 0 \Rightarrow \exists$ singular vector v in $M_{\lambda=(c,h)}$ at ~~the~~ degree $\geq -rs$



By [Hwk 3, Problem 2]: submodule of $M_{c,h}$ gen-d by v is isom to Verma $\simeq M_{c',h'}$

$$\Downarrow$$

dimension of its degree $-n$ part is $\underbrace{\geq p(n-rs)}$

$$\Downarrow$$

$\forall n \geq rs$: $\underbrace{\det_n(c,h) \vdots (h - h_{r,s}(c))^{p(n-rs)}}$

For generic c, the numbers $h_{r,s}(c)$ are pairwise distinct

$$\Downarrow$$

$$\det_n(c,h) \vdots \prod_{r,s}^{rs\leq n} (h - h_{r,s}(c))^{p(n-rs)} \xRightarrow{\text{degree reasons}} \text{differ by const}$$

$\underline{K_n}$ from Hwk.

Rmk: (a) $h_{r,r}(c)$ - polynomial in c

(b) $\underset{r\neq s}{(h-h_{r,s}(c))(h-h_{s,r}(c))}$ — polynomial in c as well

$\boxed{\text{Cor 5}}$: The Virasoro Verma module $M_{c,h}$ is irreducible iff (c,h) does not belong to:

* lines : $h - h_{r,r}(c) = 0 \iff \boxed{h + \frac{(r^2-1)(c-1)}{24} = 0}$

* quadrics : $\underbrace{(h-h_{r,s}(c))(h-h_{s,r}(c))}=0$

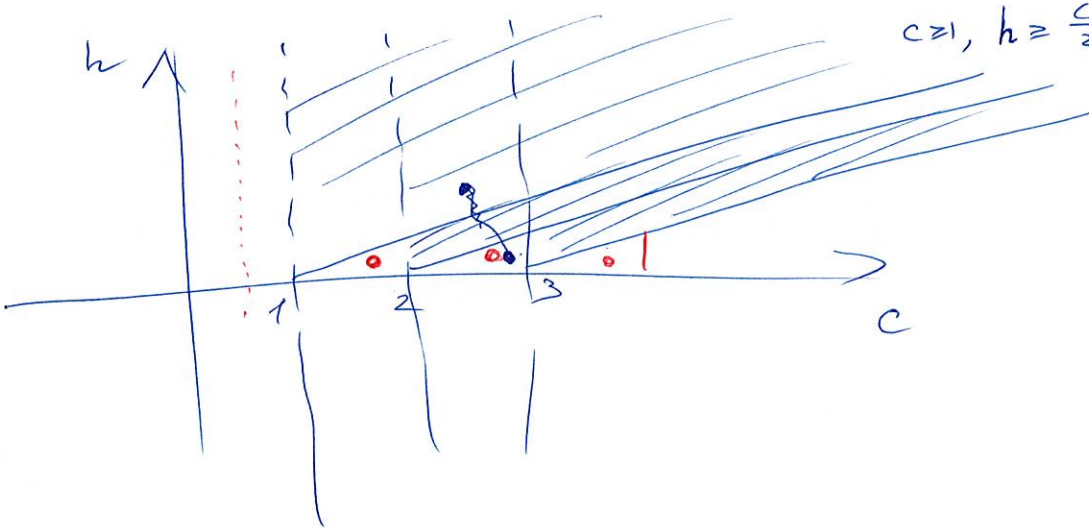
$$\boxed{\left(h-\frac{(r-s)^2}{4}\right)^2 + \frac{h}{24}(c-1)(r^2+s^2-2) + \frac{1}{576}(r^2-1)(s^2-1)(c-1)^2 + \frac{1}{48}(c-1)(r-s)^2(rs+1).}$$

$\boxed{\text{Cor 6}}$: If $h > 0$, $c > 1 \Rightarrow M_{c,h}$ - irr

i.e. $M_{c,h} \simeq L_{c,h}$

▶ Follows from exact eq-s of lines & quadrics! ■

Recall from Lecture 7, the following unitarity of $L_{h,c}$: $c \geq 1$, $h \geq \frac{c-1}{24}$ & its translations by $\mathbb{N} \times \{0\}$.



$\boxed{\text{Cor 7}}$ If $h \geq 0$, $c \geq 1 \Rightarrow L_{c,h}$ is unitary!

(proves Thm 1 from Lecture 7).

- First consider $h > 0$, $c > 1$. $\xRightarrow{\text{Cor 6}} L_{c,h} \simeq M_{c,h}$.

Know: $M_{c,h}$ - unitary when $c > 1$, $h > \frac{1}{24}$.

Matrix used to compute

Unitarity $\Longleftrightarrow$ $\det_n$ is positive def. form.

$\Downarrow$ $M_{c,h}$ - irr. in the entire region $\begin{cases} c > 1 \\ h > 0 \end{cases}$

$\det_n \neq 0 \Rightarrow$ pos. def. in $c > 1$, $h > 0$. ✓

- If $h = 0$ or $c = 1$ — get as a limit from above after passing $M_{c,h} \twoheadrightarrow L_{c,h}$.

More precisely, the limit form is non-negative, and as we pass from $M_{c,h}$ to $L_{c,h}$ we kill its kernel $\Rightarrow$ pos. def. form on $L_{c,h}$.

Q: What about region $\left\{ \begin{array}{l} 0 \le c < 1 \\ h \ge 0 \end{array} \right\}$?

A: The only unitary points (c, h) are the following discrete series

$$\boxed{\left\{ c(m),\ h_{r,s}(m) \;\middle|\; \begin{array}{l} m \in \mathbb{Z}_{\ge 0} \\ 1 \le s \le r \le m+1 \end{array} \right\}}$$

$$c(m) = 1 - \frac{6}{(m+2)(m+3)}$$

$$\Downarrow$$

We'll prove later unitarity at these points

$$h_{r,s}(m) = \frac{[(m+3)r - (m+2)s]^2 - 1}{4(m+2)(m+3)}$$

E.g. $m = 1 \Rightarrow c(m) = \frac{1}{2}$

$1 \le s \le r \le 2$

$$\left.\begin{array}{lcl} (1,1) & & h_{1,1} = 0 \\ (1,2) & \leadsto & h_{1,2} = \frac{1}{2} \\ (2,2) & & h_{2,2} = \frac{1}{16} \end{array}\right\} \longleftarrow \text{we constructed these 3 unitary rep-s long time ago!}$$