Lecture 15

03/15/2021

$\boxed{\text{Last time}}$: $\lambda = (\overset{\text{C-weight}}{c}, \overset{L_0\text{-weight}}{h}) \in \mathbb{C}^2 \rightsquigarrow M_\lambda$ - Verma over Vir

$$\boxed{\det_n(c,h) = \underbrace{K_n}_{\text{non-zero constant}} \cdot \prod_{\substack{r,s\geq 1 \\ rs\leq n}} (h - h_{r,s}(c))^{p(n-rs)} \qquad (*)}$$

det of Shapovalov/contravariant form on degree n part

where

$$h_{r,s}(c) := \frac{1}{48}\left((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}\,(r^2-s^2) - 24rs - 2 + 2c\right)$$

Rmks: (1) the RHS of $(*)$ is polynomial in h, c ($\sqrt{\ldots}$ disappear!)

(2) $\underbrace{\text{zeroes of all } \det_n}_{=\text{locus where } M_{c,h} \text{ fails to be irreducible}}$ are union of lines and quadrics

explicitly written last time

$\boxed{\text{Latter}}$: Will prove $\det_{rs}(c, h_{r,s}(c)) = 0$ (last time used as a black box)

Cor (1) If $h>0$, $c>1$ $\Rightarrow$ $M_{c,h} \simeq L_{c,h}$. (i.e. $M_{(c,h)}$ is irreducible)

Verma

(2) If $h \geq 0$, $c \geq 1$ $\Rightarrow$ $L_{c,h}$ – unitary.

! This leaves open the question of when $L_{c,h}$ is unitary for $h \geq 0$ & $0 \leq c < 1$.

Claim For $0 \leq c < 1$, $h \geq 0$:

$L_{c,h}$ – unitary $\Longleftrightarrow$ $\exists\, m \in \mathbb{Z}_{\geq 0}$ $\exists$ $1 \leq s \leq r \leq m+1$ s.t.

$$c = c(m) = 1 - \frac{6}{(m+2)(m+3)}$$

$$h = h_{r,s}(c) = \frac{[(m+3)r - (m+2)s]^2 - 1}{4(m+2)(m+3)}$$

We'll prove "$\Longleftarrow$" latter on!

$\boxed{\text{Prop 1}}$: (a) If $c=0$, then $L_{c,h}$ is unitary iff $h=0$.

(b) $L_{0,h} = M_{0,h}$ iff $h \neq \frac{m^2-1}{24}$ for $m \in \mathbb{Z}$.

(c) $L_{1,h} = M_{1,h}$ iff $h \neq \frac{m^2}{4}$ for $m \in \mathbb{Z}$.

(c) Looking at explicit eq-s of lines & quadrics from last time we see that $M_{1,h}$ is irreducible iff:

* $h \neq 0$ (coming from eq-s of lines) $\leftarrow h + \frac{(r^2-1)\overbrace{(c-1)}^{=0}}{24} = 0$
* $h \neq \frac{(r-s)^2}{4}$ for $r \neq s$ (coming from eq-s of quadrics)

$\Rightarrow h \neq \frac{m^2}{4}$. $\forall m \in \mathbb{Z}$

(b) $\underset{\substack{\uparrow \\ c=0}}{h_{r,s}(0)} = \frac{(3r-2s)^2-1}{24}$ $\forall r,s \geq 1$. $\Longrightarrow$ $M_{0,h}$ - irred iff $h \neq \frac{m^2-1}{24}$ $\forall m \in \mathbb{Z}$.

($3r-2s$ can take any $\mathbb{Z}$-value)

Part (a) is a bit trickier, see next page !

(a) $$\det\begin{pmatrix} (L_{-N}^2 v_\lambda, L_{-N}^2 v_\lambda) & (L_{-N}^2 v_\lambda, L_{-2N} v_\lambda) \\ (L_{-2N} v_\lambda, L_{-N}^2 v_\lambda) & \underbrace{(L_{-2N} v_\lambda, L_{-2N} v_\lambda)} \end{pmatrix} = ?$$

Here: $(\cdot,\cdot)$ - contrav. form on M_λ

$N \in \mathbb{N}$.

$$\overbrace{L_{-N}^2 v_\lambda, \quad L_{-2N} v_\lambda}^{\text{lin. indep. vectors (both } M_\lambda^+ \,\&\, L_\lambda^+\text{, with only 1 possible exception of } N)} \in M_\lambda^+[-2N].$$

$$\Downarrow$$

$$\in L_\lambda^+[-2N].$$

- $(L_{-2N} v_\lambda, L_{-2N} v_\lambda) = (v_\lambda, L_{2N} L_{-2N} v_\lambda) = (v_\lambda, (L_{-2N} L_{2N} + 4N\cdot L_0 + \frac{8N^3-2N}{12} C) v_\lambda)$

$$= 4N\cdot h + \frac{8N^3-2N}{12}\cdot \cancel{c} \xrightarrow{\;c=0\;} = \underline{4Nh}$$

- $(L_{-2N} v_\lambda, L_{-N}^2 v_\lambda) = (v_\lambda, L_{2N} L_{-N} L_{-N} v_\lambda) = (v_\lambda, (L_{-N} L_{2N} L_{-N} + 3N\cdot L_N L_{-N}) v_\lambda)$

$$= 3N\left(2N\cdot h + \frac{N^3-N}{12} c\right) + 0 = 6N^2\cdot h + \frac{3N(N^3-N)}{12}\cdot \cancel{c} \text{ as } c=0. \quad = \underline{6N^2 h}$$

$L_{-N} L_{2N} + 3N L_N$ ← both annihilate v_λ!

- $(L_{-N}^2 v_\lambda, L_{-N}^2 v_\lambda) \overset{\text{Exercise}}{=} 8N^2h^2 + 4N^3 h$

SO: $\det = \det\begin{vmatrix} 8N^2h^2 + 4N^3h & 6N^2h \\ 6N^2h & 4Nh \end{vmatrix} = 32N^3h^3 + 16N^4h^2 - 36N^4h^2 = \underbrace{32N^3h^3 - 20N^4h^2}_{<0 \text{ for } N\gg 1 \text{ unless } h=0!}.$

BUT: For $h=0$: $L_{0,0}$ = trivial ⇒ unitary!

$\boxed{\text{Today}}$ From $\overline{\mathfrak{a}}_\infty$ to affine Lie algebras. ← Follows §9 of [Kac-Raina]

- Loop algebra $L\mathfrak{gl}_n = \mathfrak{gl}_n[t,t^{-1}] = \mathfrak{gl}_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$.

Its elements: $\boxed{a(t) = \sum_{k\in\mathbb{Z}} a_k t^k, \ a_k \in \mathfrak{gl}_n \text{ and only finitely many are } \neq 0}$

Basis: $\{E_{ij}(k) = E_{ij}\cdot t^k \mid 1\leq i,j\leq n, \ k\in\mathbb{Z}\}$

Lie bracket: $[E_{ij}(k), E_{i'j'}(k')] = \delta_{i'j}\cdot E_{ij'}(k+k') - \delta_{ij'} E_{i'j}(k+k')$.

- $\mathfrak{gl}_n \curvearrowright \mathbb{C}^n = \begin{pmatrix} \cdot \\ \vdots \\ \cdot \end{pmatrix} \updownarrow n \rightsquigarrow L\mathfrak{gl}_n \curvearrowright \mathbb{C}^n[t,t^{-1}]$

↑ choose standard basis: $e_1, e_2, \ldots, e_n$ so that $E_{ij}(e_{j'}) = \delta_{jj'}\cdot e_i$

↑ basis: $\{e_i\cdot t^k \mid 1\leq i\leq n, \ k\in\mathbb{Z}\}$

So: $\boxed{E_{ij}(k)(e_{j'}\cdot t^{k'}) = \delta_{jj'}\cdot e_i\cdot t^{k+k'}.}$ (**)

• Denote : $\boxed{v_{i-kn} := e_i \cdot t^k}$ $\quad \{(1\le i\le n,\ k\in\mathbb{Z})\} \xleftrightarrow{1\text{-to-}1} \mathbb{Z}$.

$$(i,k) \longleftrightarrow i-kn.$$

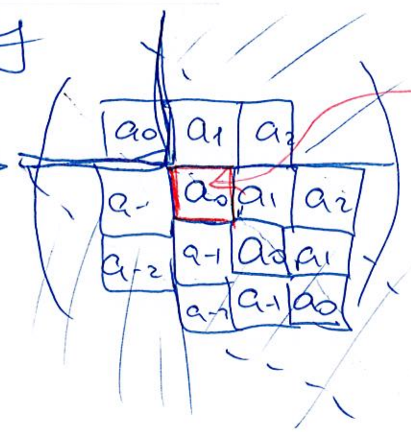
Note : (**) $\boxed{E_{ij}(k)\left(v_{j'+nk'}\right) = \delta_{jj'} \cdot v_{i+n(k'-k)} \quad \forall\ 1\le i,j,j'\le n \ \forall k,k'\in\mathbb{Z}}$

Recall : Also had ∞-dim space V with basis $\{v_i\}_{i\in\mathbb{Z}}$ when discussing gl_∞, $\bar{a}_\infty$, a_∞. } We shall identify $v_j\in V$ with the above v_j's!

THUS:

$L gl_n \xhookrightarrow[\text{Lie alg. embedding}]{\tau} \bar{a}_\infty$ explicitly given by

$$\sum_{k\in\mathbb{Z}} a_k t^k \longmapsto$$



at the intersection of rows & columns #1, 2, …, n.

Explicitly : $\boxed{\tau\left(E_{ij}(k)\right) = \sum_{m\in\mathbb{Z}} E_{nm+i,\ n(m+k)+j}}$

Rmks: (a) τ is compatible with multiplication.

(b) $X t^k \xmapsto{\ \omega\ } X^{\dagger} \cdot t^{-k}$ – anti-linear anti-involution, $X^{\dagger}$ = Hermitian adjoint of $X \in \mathfrak{gl}_n$

Easy Claim: $\boxed{\tau(\omega(a(t))) = (\tau(a(t)))^{\dagger}}$ ← Hermitian adjoint of $\mathbb{Z}\times\mathbb{Z}$ matrix.

(c) $$\tau\underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ t & & & 0 \end{pmatrix}}_{E_{12}+E_{23}+\dots+E_{n-1,n}+t\cdot E_{n1}\,\in\, L\mathfrak{gl}_n} = \tau\left(\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} + t\cdot\begin{pmatrix} & & \\ & 0 & \\ 1 & & \end{pmatrix}\right) = \underset{\overline{\mathfrak{a}}_\infty}{\overset{T}{\cap}}$$

Recall: $T = \sum_{i\in\mathbb{Z}} E_{i,i+1}$

$$\begin{array}{ccc} L\mathfrak{gl}_n & \overset{\tau}{\hookrightarrow} & \overline{\mathfrak{a}}_\infty \\ \Big\downarrow & & \Big\downarrow \text{Japanese 2-cocycle } \alpha \\ \text{central extension of } L\mathfrak{gl}_n & \hookrightarrow & \mathfrak{a}_\infty = \overline{\mathfrak{a}}_\infty \oplus \mathbb{C}\cdot K \end{array}$$

Recall:

$\alpha : \overline{\mathfrak{a}}_\infty \times \overline{\mathfrak{a}}_\infty \to \mathbb{C}$.

rows & columns of A_{11}: $0,-1,-2,\dots$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\alpha(A,B) = \mathrm{Tr}(A_{12}B_{21} - B_{12}A_{21})$$

The next Lemma will show that the above central extension of $L\mathfrak{gl}_n$ exactly coincides with the one from our Week 1 of classes!

$\boxed{\text{Lemma 1}}$

$$\underbrace{\alpha_\tau(a(t), b(t))}_{\alpha(\tau(a(t)),\, \tau(b(t)))} = \operatorname{Res}_{t=0} \big(a'(t), b(t)\big)dt \overset{\text{Explicitly}}{=} \sum_{k\in\mathbb{Z}} k \cdot \operatorname{Tr}(a_k b_{-k})$$

$a(t) = \sum a_k t^k$
$b(t) = \sum b_k t^k$.

Easy Exercise

$$\alpha_\tau\big(E_{ij}(k), E_{i'j'}(k')\big) = k \cdot \delta_{ij'} \cdot \delta_{i'j}\, \delta_{k+k',0}$$

$$\Downarrow$$

$$\alpha_\tau(at^k, bt^{k'}) = k \cdot \operatorname{Tr}(ab) \cdot \delta_{k+k',0}$$

∎

$\boxed{\text{UPSHOT}}$ $\boxed{\widehat{gl}_n = Lgl_n \oplus \mathbb{C}\cdot K \hookrightarrow \underset{\overline{\sigma}_\infty \oplus \mathbb{C}\cdot K}{\overset{\shortparallel}{\sigma_\infty}}}$ (***)

! Rmk: When $n=1 \Rightarrow \boxed{\widehat{gl}_1 \simeq \mathcal{A}}$ & $\boxed{\mathcal{A} \hookrightarrow \sigma_\infty \text{ - what we constructed before!}}$

$\boxed{\text{So}}$: We can regard (***) as a "higher rank" generalization of previously established $\mathcal{A} \hookrightarrow \sigma_\infty$.

NOTE: $sl_n \hookrightarrow gl_n \rightsquigarrow Lsl_n \rightsquigarrow \widehat{sl}_n := Lsl_n \oplus \mathbb{C}\cdot K$

$$\begin{array}{ccc} Lsl_n & & \widehat{sl}_n \\ \downarrow & & \downarrow \\ Lgl_n & & \widehat{gl}_n \hookrightarrow \sigma_\infty \end{array}$$

$$\begin{array}{c}\widehat{\mathfrak{gl}}_n \\ \cup \\ \widehat{\mathfrak{sl}}_n\end{array} \hookrightarrow \mathfrak{a}_\infty \curvearrowright \mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)} \quad \forall m \in \mathbb{Z}$$

$\boxed{\text{Cor:}}$ $\mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)}$ become modules over $\widehat{\mathfrak{gl}}_n \supset \widehat{\mathfrak{sl}}_n$ of level 1 !

i.e. K acts by Id.

- $(\cdot|\cdot)$ - form on $\mathfrak{gl}_n$ given by $(X|Y) = Tr(XY)$ - symmetric, nondeg, inv. form on $\mathfrak{gl}_n$

$(\cdot|\cdot)$ - on $L\mathfrak{gl}_n$ given by $(a(t) \mid b(t)) = Res_{t=0} Tr(a(t) \cdot b(t)) \frac{dt}{t}$ - symm. nondeg. inv. bil. form on $L\mathfrak{gl}_n$

extend to $\widehat{\mathfrak{gl}}_n$

$(\cdot|\cdot)$ on $\widehat{\mathfrak{gl}}_n$ is defined by declaring $\boxed{(K| L\mathfrak{gl}_n) = 0, \ (K|K) = 0}$

needed for inv. of our pairing.

WARNING: degenerate pairing as K is in its kernel!

To fix this, we'll add one more generator!

<u>Note</u>: To apply our general machinery of $\mathbb{Z}$-graded non-deg. Lie algebras we wish to have a non-deg. pairing!

Define

- $d : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ - derivation

$$K \mapsto 0 \qquad \underset{\mathfrak{gl}_n \text{ or } \mathfrak{sl}_n}{X} \cdot t^k \mapsto k \cdot X \cdot t^k$$
$$a(t) \mapsto t a'(t) ,$$

|| Def: $\boxed{\tilde{\mathfrak{g}} := \mathbb{C} d \ltimes \hat{\mathfrak{g}}}$

here we now choose $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{gl}_n$ but latter will choose any simple f.d. Lie alg. $\mathfrak{g}$.

$$[d, K] = 0$$
$$[d, Xt^k] = k \cdot Xt^k .$$

Lemma 2: Extending the above pairing on $\hat{\mathfrak{gl}}_n$ to that on $\tilde{\mathfrak{gl}}_n$ via

$$\boxed{(d|d) = 0, \quad (d|a(t)) = (a(t)|d) = 0, \quad (d|K) = (K|d) = 1}$$

gives rise to a symm. nondeg. inv. pairing on $\tilde{\mathfrak{gl}}_n$

(1) $(\underbrace{[d, Xt^k]}_{k \cdot Xt^k} | yt^l) \overset{?}{=} -(Xt^k | \underbrace{[d, yt^l]}_{l \cdot yt^l})$

- both sides vanish if $k+l \neq 0$
- If $k+l = 0 \Rightarrow k = -l \Rightarrow$ obvious equality.

(2) $\underbrace{(\underbrace{[Xt^k, d]}_{-k \cdot Xt^k} | yt^l)}_{-k(X|y) \cdot \delta_{k+l,0}} \overset{?}{=} \underbrace{-(d | [Xt^k, yt^l])}_{-k \cdot \delta_{k+l,0} \cdot (X|y) \cdot \underbrace{(d|K)}_{1}}$

$$-k(X|y) \cdot \delta_{k+l,0} = -k \cdot \delta_{k+l,0} \cdot (X|y) \cdot \underbrace{(d|K)}_{1}$$

All other verifications are immediate!

$$\begin{array}{ccccccc} \mathfrak{gl}_n & \rightsquigarrow & L\mathfrak{gl}_n & \rightsquigarrow & \widehat{\mathfrak{gl}}_n & \rightsquigarrow & \widetilde{\mathfrak{gl}}_n \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid \\ \mathfrak{sl}_n & \rightsquigarrow & L\mathfrak{sl}_n & \rightsquigarrow & \widehat{\mathfrak{sl}}_n & \rightsquigarrow & \widetilde{\mathfrak{sl}}_n \end{array}$$

affine Lie alg-s.

• $\widehat{\mathfrak{sl}}_n, \widehat{\mathfrak{gl}}_n, \underline{\underline{\widetilde{\mathfrak{sl}}_n}}, \widetilde{\mathfrak{gl}}_n$ – $\mathbb{Z}$-graded Lie algebras via the "principal gradation".

e.g. $\widetilde{\mathfrak{sl}}_n = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_-$, where: $\tilde{\mathfrak{n}}_+ = \mathfrak{n}_+ + \sum_{k>0} t^k \cdot \mathfrak{sl}_n$, $\mathfrak{n}_+ \subset \mathfrak{sl}_n$ – strictly upper-Δ

triangular decomp. of $\mathbb{Z}$-gr. Lie alg

$\tilde{\mathfrak{n}}_- = \mathfrak{n}_- + \sum_{k>0} t^{-k} \cdot \mathfrak{sl}_n$, $\mathfrak{n}_- \subset \mathfrak{sl}_n$ – strictly lower-Δ

$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$, $\mathfrak{h} \subset \mathfrak{sl}_n$ – diagonal

• $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_{\mathfrak{sl}_n}$ has a basis $\left\{ h_i = E_{ii} - E_{i+1,i+1} \ (1 \le i \le n-1),\ h_0 = \underbrace{K - (h_1 + \ldots + h_{n-1})}_{K - (E_{11} - E_{nn})},\ d \right\}$

$h_i = [\underbrace{E_{i,i+1}}_{e_i}, \underbrace{E_{i+1,i}}_{f_i}]$ $h_0 = [\underbrace{E_{n1} \cdot t}_{e_0}, \underbrace{E_{1n} t^{-1}}_{f_0}]$

$\boxed{\text{Def 1}}$: The el-s $\{\tilde{\omega}_i\}_{i=0}^{n-1} \subset \tilde{\mathfrak{h}}^*_{\mathfrak{sl}_n}$ are defined via: $\boxed{\begin{array}{c} \tilde{\omega}_i(h_j) = \delta_{ij},\ \tilde{\omega}_i(d) = 0 \quad (0 \le j \le n-1) \\ \Downarrow \\ \tilde{\omega}_i(K) = 1. \end{array}}$

Def 2 The el-s $\{\tilde{\omega}_m\}_{m\in\mathbb{Z}} \subset (\tilde{\mathfrak{h}}_{gl_n})^* = \mathrm{span}(\langle E_{11},\dots,E_{nn},K,d\rangle)^*$ are defined via:

$$\boxed{\tilde{\omega}_m(d)=0,\quad \tilde{\omega}_m(K)=1,\quad \tilde{\omega}_m(E_{ii})=\begin{cases}1, & \text{if } i\le \bar{m}\\ 0, & \text{if } i>\bar{m}\end{cases}+\frac{m-\bar{m}}{n},\qquad \bar{m}:=m \bmod n\in\{0,1,\dots,n-1\}.}$$

! **Rmk:** $\forall m\in\mathbb{Z}$, the restriction of $\tilde{\omega}_m$ to $\tilde{\mathfrak{h}}_{sl_n} \subset \tilde{\mathfrak{h}}_{gl_n}$ equals $\tilde{\omega}_{\bar{m}}$.

$$\widehat{gl}_n \overset{(1)}{\curvearrowright} \mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)}$$

$$\downarrow$$

$$\widetilde{gl}_n = \widehat{gl}_n \rtimes \mathbb{C}d.$$

Lemma 3 (Hwt Problem): $\exists!$ unique extension of (1) to $\widetilde{gl}_n \curvearrowright \mathcal{F}^{(m)}$ s.t. $d(\psi_m) = 0$.

$\boxed{\text{Prop 2}}$: For any $m \in \mathbb{Z}$, the $\widetilde{gl}_n$-module $\mathcal{F}^{(m)}$ is irreducible with h. wt $= \widetilde{\omega}_m$.

- Irreducibility: $\tau\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ t & & 0 & 0 \end{pmatrix} = T \Rightarrow \tau\left(\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ t & & 0 & 0 \end{pmatrix}^k\right) = T^k \leftarrow$ generate $\mathcal{A} \subseteq \mathfrak{a}_\infty$.

$\mathcal{A} \hookrightarrow \widehat{gl}_n \hookrightarrow \mathfrak{a}_\infty.$

$\boxed{\text{Know}}$: $\mathcal{A} \curvearrowright \mathcal{F}^{(m)}$ is irred $\Rightarrow \widehat{gl}_n \curvearrowright \mathcal{F}^{(m)}$ - irred. $\Rightarrow \widetilde{gl}_n \curvearrowright \mathcal{F}^{(m)}$ - irred.

NOTE: We actually see that $\mathcal{F}^{(m)}$ is irred. as $\widehat{gl}_n$-module.

- $\tilde{n}_+ \xhookrightarrow{\tau} \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$ - upper-Δ matrices in $\bar{a}_\infty$ which annihilate ψ_m

$$\Rightarrow \boxed{\tilde{n}_+(\psi_m) = 0.}$$

- Remarks : $\forall h \in \tilde{\eta}_{gl_n}$: $\boxed{h(\psi_m) \overset{?}{=} \tilde{\omega}_m(h) \cdot \psi_m.}$

  - $h = d$: $0 = 0$ ✓
  - $h = K$: $\psi_m = \psi_m$ ✓

- $h = E_{ii} \Rightarrow \boxed{\tau(E_{ii}) = \sum_{\substack{j \in \mathbb{Z} \\ j \equiv_n i}} E_{jj} \in \bar{a}_\infty}$ & $\boxed{\hat{\rho}(E_{jj})\psi_m = (\delta_{j \leq m} - \delta_{j \leq 0}) \cdot \psi_m.}$

So:

* $m > 0$: $\hat{\rho}(\tau(E_{ii}))\psi_m = \underbrace{\#\{0 < j \leq m \mid j \equiv_n i\}}_{\tilde{\omega}_m(E_{ii})} \cdot \psi_m = \tilde{\omega}_m(E_{ii}) \cdot \psi_m$ ✓

* $m \leq 0$: analogous .... Exercise!

$\boxed{\text{Outcome}}$ $\mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)}$ is irred. $\widehat{\mathfrak{gl}}_n$ - module
& $\mathfrak{gl}_n$ - module.

$\boxed{Q}$ What about $\widehat{\mathfrak{sl}}_n, \mathfrak{sl}_n \curvearrowright \mathcal{F}^{(m)}$?

! $\mathcal{F}^{(m)}$ is <u>not irred.</u> as $\widehat{\mathfrak{sl}}_n$ - module !

▸ Consider $\underbrace{\{T^{ni}\}_{i\in\mathbb{Z}}}$ $\xleftrightarrow{\tau}$ $t^i \cdot \underbrace{\mathrm{Id}}_{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathfrak{gl}_n}$ ← in the center of $L\mathfrak{gl}_n$

in the image $\tau(L\mathfrak{gl}_n)$

but <u>not</u> in the image $\tau(L\mathfrak{sl}_n)$.

$\Downarrow$

they commute with $L\mathfrak{sl}_n$.

$\Downarrow$

$\boxed{\text{commute with } \widehat{\mathfrak{sl}}_n}$

But T^{ni} don't act ~~trivially~~ by scalar mult. on $\mathcal{F}^{(m)}$

$\Downarrow$ by Schur Lemma

$\mathcal{F}^{(m)}$ cannot be irred. $\widehat{\mathfrak{sl}}_n$ - mod. ∎

- $\underbrace{\mathcal{A}}_{\text{oscillator algebra}} \supset \underbrace{\mathcal{A}^{(n)}}$

oscillator algebra $\{a_m\}_{m\in\mathbb{Z}} \cup \{K\}$

gen-d by $\{a_{ni}\}_{i\in\mathbb{Z}} \cup \{K\}$.

$\boxed{\text{Lemma 4}}$: $\mathcal{A} \xrightarrow[\sim]{\text{Lie alg. isom.}} \mathcal{A}^{(n)}$

(Obvious)

$a_i \longmapsto a_{ni}$

$K \longmapsto nK.$

$\boxed{\text{Lemma 5}}$ (Hwk Problem): The Lie alg $\hat{gl}_n$ and $(\hat{sl}_n \oplus \mathcal{A}^{(n)})/(K_1 - K_2)$ are isom.

$K_1 = (K, 0)$, $K_2 = (0, K)$