03/23/2021

Lecture 16

Last time:

• $\boxed{L\mathfrak{gl}_n \overset{\tau}{\hookrightarrow} \bar{\mathfrak{a}}_\infty}$

$$\sum_{k\in\mathbb{Z}} a_k t^k \overset{\tau}{\longmapsto} \begin{pmatrix} \ddots & & & & \\ & a_0 & a_1 & a_2 & \\ & a_{-1} & a_0 & a_1 & \\ & & a_{-1} & a_0 & a_1 \\ & & & & \ddots \end{pmatrix}$$

$a_k \in \mathfrak{gl}_n$

columns & rows $\# 1, 2, \dots, n$

$\mathfrak{gl}_n \curvearrowright \mathbb{C}^n$

$\rightsquigarrow$

$L\mathfrak{gl}_n \curvearrowright \underbrace{\mathbb{C}^n[t,t^{-1}]}$

basis $e_i \otimes t^k \leftrightarrow \{v_j\}$ - basis of V.

$$\tau \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ t & 0 & & 0 \end{pmatrix} = T = \sum_{i\in\mathbb{Z}} E_{i,i+1}$$

$\Downarrow$

$$\boxed{\tau \begin{pmatrix} t^k & & 0 \\ & \ddots & \\ 0 & & t^k \end{pmatrix} = T^{nk}}$$

$\longleftarrow$ Will be needed for today.

• $\bar{\mathfrak{a}}_\infty \hookrightarrow \mathfrak{a}_\infty$ - central extension via the Japanese 2-cocycle α

$A, B \in \bar{\mathfrak{a}}_\infty \Rightarrow \boxed{\alpha(A,B) = \mathrm{Tr}(A_{12}B_{21} - B_{12}A_{21})}$

rows & columns $\# 0, -1, -2, \dots$ $\longrightarrow$ $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ $\rightsquigarrow$ $\boxed{\widehat{\mathfrak{gl}}_n \hookrightarrow \mathfrak{a}_\infty}$

$\widehat{\mathfrak{gl}}_n = L\mathfrak{gl}_n \oplus \mathbb{C}\cdot K$

$\mathfrak{a}_\infty \curvearrowright \mathcal{F}^{(m)}$

$\rightsquigarrow$

$\boxed{\widehat{\mathfrak{gl}}_n \curvearrowright \mathcal{F}^{(m)}}$

- $\underbrace{\text{Need}}_{\text{for the non-deg. pairing}}$ extra generator d, s.t. $(d \mid L\mathfrak{gl}_n) = 0$
  $(d \mid K) = 1$.

$\exists!$ derivation $d : \widehat{\mathfrak{gl}}_n \circlearrowleft$, s.t. $K \mapsto 0$, $xt^k \mapsto k \cdot xt^k$ $\quad \forall k \in \mathbb{Z}$, $\forall x \in \mathfrak{gl}_n$.

$$\boxed{\widetilde{\mathfrak{gl}}_n := \mathbb{C}d \ltimes \widehat{\mathfrak{gl}}_n}$$

Down-to-earth: $\widetilde{\mathfrak{gl}}_n$ is obtained from $\widehat{\mathfrak{gl}}_n$ by adding one new generator d, s.t. $\boxed{[d, K] = 0, \quad [d, xt^k] = k \cdot xt^k.}$

* Then: $\exists!$ non-deg. inv. pairing on $\widetilde{\mathfrak{gl}}_n$.

* Hwk 8: $\widehat{\mathfrak{gl}}_n \curvearrowright \mathcal{F}^{(m)}$ uniquelly extends to $\widetilde{\mathfrak{gl}}_n \curvearrowright \mathcal{F}^{(m)}$, s.t. $d(\psi_m) = 0$.

• $\boxed{\text{Prop 1}}$: $\widehat{\mathfrak{gl}}_n \curvearrowright \mathcal{F}^{(m)}$ - irreducible repr. (already irred. as $\overline{\mathfrak{gl}}_n$-module) with h.wt $\underbrace{\tilde{\omega}_m}$

↑ defined last time $\left(\begin{array}{l} \text{need to specify:} \\ \tilde{\omega}_m(d)=0,\ \tilde{\omega}_m(K)=1,\ \tilde{\omega}_m(E_{ii})=\dots \end{array} \right)$

<u>Note</u> :

$$\left.\begin{array}{lll}
\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}\cdot I & & \\
\quad\Downarrow & & \\
L\mathfrak{sl}_n \hookrightarrow L\mathfrak{gl}_n \oplus \bigoplus_{k\in\mathbb{Z}} \mathbb{C}\cdot\underbrace{(\mathrm{Id}\cdot t^k)}_{\begin{pmatrix} t^k & 0 \\ 0 & t^k \end{pmatrix}} & & \\
\quad\Downarrow & & \\
\widehat{\mathfrak{sl}}_n \hookrightarrow \widehat{\mathfrak{gl}}_n & -//- & \\
\quad\Downarrow & & \\
\overline{\mathfrak{sl}}_n \hookrightarrow \overline{\mathfrak{gl}}_n & -//- &
\end{array}\right\}$$

$\widehat{\mathfrak{sl}}_n \curvearrowright \mathcal{F}^{(m)}$, $\overline{\mathfrak{sl}}_n \curvearrowright \mathcal{F}^{(m)}$ — $\boxed{Q}$: Are they irreducible?

$\boxed{\text{Last time}}$ $\widehat{\mathfrak{sl}}_n \curvearrowright \mathcal{F}^{(m)}$ is <u>not</u> <u>irreducible</u>!

<u>Reason</u>: $\begin{pmatrix} t^k & & 0 \\ & \ddots & \\ 0 & & t^k \end{pmatrix}$ commute with $\underbrace{\widehat{\mathfrak{sl}}_n}_{\sum a_k t^k \oplus j\cdot K}$ (both viewed as being inside $\overline{\mathfrak{gl}}_n$)

$\updownarrow \tau$

$T^{nk} = \sum_{i\in\mathbb{Z}} E_{i,i+nk}$.

$\xrightarrow[\text{Schur Lemma}]{T^{nk}\text{ acts nontrivially on } \mathcal{F}^{(m)}}$ $\boxed{\widehat{\mathfrak{sl}}_n \curvearrowright \mathcal{F}^{(m)} \text{ is not irred}}$

Recall:

$$\widehat{\mathfrak{gl}}_n \supset \widehat{\mathfrak{sl}}_n$$

$$\cup \quad \bigoplus_{k\in\mathbb{Z}} \mathbb{C}\cdot \underbrace{(t^k \mathrm{Id})}_{\begin{pmatrix} t^k & & 0 \\ & \ddots & \\ 0 & & t^k \end{pmatrix}} \rightsquigarrow \text{generate a copy } \overset{\{a_{ni}\}_{i\in\mathbb{Z}}\cup\{K\} \text{ basis}}{\mathcal{A}^{(n)}} \subseteq \mathcal{A} \leftarrow \text{basis } \underbrace{\{a_i\}_{i\in\mathbb{Z}}\cup\{K\}}$$

$$\mathcal{A}^{(n)} \simeq \mathcal{A}$$

* **Hwk 8**: $\widehat{\mathfrak{gl}}_n \simeq \left(\widehat{\mathfrak{sl}}_n \oplus \mathcal{A}^{(n)}\right) \Big/ \underbrace{(K_1 - K_2)}_{K_1 = (K,0),\ K_2 = (0,K)}$

i.e. we identify two central el-s!

$$\mathcal{F}^{(m)} \simeq \mathcal{B}^{(m)} = \mathbb{C}[x_1, x_2, x_3, \ldots]$$

Def: $\mathcal{B}_0^{(m)} := \{p \in \mathcal{B}^{(m)} \mid T^{ni}(p) = 0 \ \forall i>0\}$

$= \{p \in \mathcal{B}^{(m)} \mid \partial_{x_{ni}}(p) = 0 \ \forall i>0\} \simeq \mathbb{C}[\{x_j\}_{j \nmid n}]$.

$\boxed{\text{Prop 2}}$: (1) For $m \in \mathbb{Z}$, $\mathcal{B}^{(m)}_0$ is an irreducible $\widehat{sl}_n$-module
($\Rightarrow$ irred. $\widetilde{sl}_n$-module)

with the h.wt $= \underbrace{\widetilde{\omega}_{\bar{m}}}_{\text{last time}}$, $\underset{\{0,1,\dots,n-1\}}{\bar{m}} := m \mod n$.

(2) $\mathcal{B}^{(m)} \simeq \mathcal{B}^{(m)}_0 \otimes \widetilde{F}_m$ ← Fock space

$\widehat{gl}_n \circlearrowright \mathcal{B}^{(m)}$, $\widehat{sl}_n \circlearrowright \mathcal{B}^{(m)}_0$, $\mathcal{A}^{(n)} \circlearrowright \widetilde{F}_m$

← isomorphism of $\widehat{sl}_n \oplus \mathcal{A}^{(n)}$-modules

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- Set $\widetilde{F}_m = \mathbb{C}[x_n, x_{2n}, x_{3n}, \dots] \simeq$ Fock module of $\mathcal{A} \simeq \mathcal{A}^{(n)}$.

Then: $\mathcal{B}^{(m)} = \mathbb{C}[\{x_j\}_{j\geq 1}] = \mathbb{C}[\{x_j\}_{j \nmid n}] \otimes \mathbb{C}[\{x_j\}_{j \mid n}] = \mathcal{B}^{(m)}_0 \otimes \widetilde{F}_m$

$\widehat{sl}_n \circlearrowright \mathcal{B}^{(m)}_0$, $\mathcal{A}^{(n)} \circlearrowright \widetilde{F}_m$

Prop 1: $\widehat{gl}_n \curvearrowright \mathcal{B}^{(m)}$ is irred $\Longrightarrow \widehat{sl}_n \curvearrowright \mathcal{B}^{(m)}_0$ is irreducible!

- $\omega_m \in (\widetilde{\mathfrak{h}}_{gl_n})^* \twoheadrightarrow (\widetilde{\mathfrak{h}}_{sl_n})^* \ni \widetilde{\omega}_{\bar{m}}$

$\omega_m \mapsto \widetilde{\omega}_{\bar{m}}$

Thm 1 The highest weight $\mathfrak{sl}_n$-repr. L_λ is unitary iff

(*) $$\boxed{\lambda = k_0\tilde{\omega}_0 + k_1\tilde{\omega}_1 + \_\_ + k_{n-1}\tilde{\omega}_{n-1} \quad \text{with all } k_i \in \mathbb{Z}_{\geq 0}}$$

- $L_{\tilde{\omega}_m} \underset{\text{Prop2}}{\overset{\mathfrak{sl}_n\text{-mod}}{\simeq}} \mathcal{F}^{(m)}$ – unitary as $\mathfrak{a}_\infty$–mod (with all $\frac{\infty}{2}$-wedges forming an orthogonal basis) $(0 \leq m \leq n-1)$

$\Downarrow$

also unitary as $\mathfrak{sl}_n$-mod.

$\Downarrow$

- If λ is as in (*), then

$L_{\tilde{\omega}_0}^{\otimes k_0} \otimes L_{\tilde{\omega}_1}^{\otimes k_1} \otimes \_\_ \otimes L_{\tilde{\omega}_{n-1}}^{\otimes k_{n-1}}$ – unitary

Ψ

Consider: $v := \otimes$ h.wt. vectors – singular vector of weight λ

$\rightsquigarrow$

consider $\mathfrak{sl}_n$-submodule V generated by $\mathfrak{sl}_n \curvearrowright v$

Then:
- V – unitary
- V – h.wt. of wt λ

$\Rightarrow V \simeq L_\lambda \Rightarrow \boxed{L_\lambda\text{– unitary!}} \Rightarrow$ Proved "if" part

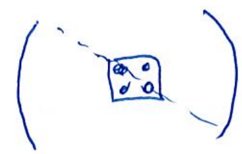
One of the common spots where we crucially use unitarity

Opposite way : $\boxed{\text{the "only if" part}}$?   Idea: Restrict to several sl_2's sitting inside.

Know: L_λ - unitary $\Longrightarrow$ λ is of the form $(*)$.

Recall: For sl_2, the irred. h.wt. module $\underbrace{L_{\lambda'}}_{\lambda' \in \mathbb{C}}$ is unitary iff $\lambda' \in \mathbb{Z}_{\geq 0}$,

$sl_2 \overset{?}{\hookrightarrow} \widehat{sl}_n$



Consider : $sl_2 \overset{\iota^{(i)}}{\hookrightarrow} sl_n \subset \widehat{sl}_n$

$e \longmapsto E_{i,i+1}$

$f \longmapsto E_{i+1,i}$

$h \longmapsto E_{ii} - E_{i+1,i+1} = h_i$

$\widehat{sl}_n \curvearrowright L_\lambda \ni v_\lambda^+$ - unitary

$\uparrow \iota^{(i)}$ $\curvearrowright$ - unitary

sl_2

$\boxed{e(v_\lambda^+) = 0.}$ $\underbrace{\iota^{(i)}(h)}_{\underbrace{E_{ii} - E_{i+1,i+1}}_{h_i}}(v_\lambda^+) = \lambda(h_i) \cdot v_\lambda^+$

So: $\boxed{\forall\ 1 \leq i \leq n-1:\ \lambda(h_i) \in \mathbb{Z}_{\geq 0}}$

Need also one more embedding $sl_2 \hookrightarrow \widehat{sl}_n$ to get $\lambda(h_0) \in \mathbb{Z}_{\geq 0}$

(which does not factor through $sl_n \subset \widehat{sl}_n$)

Consider: $\mathfrak{sl}_2 \xhookrightarrow{\iota^{(0)}} \widehat{\mathfrak{sl}}_n$ ⟵ Lie alg. homom.

$$e \longmapsto t\cdot E_{n1} = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

$$f \longmapsto t^{-1}\cdot E_{1n} = \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix}$$

$$h \longmapsto K + \underbrace{E_{nn} - E_{11}} = h_0$$

Exercise: Verify $\begin{cases} [e,f] = h \\ [h,e] = 2e \\ [h,f] = -2f \end{cases}$ are compatible with these assignments

$$L_\lambda \ni v_\lambda^+ \text{ - killed by } \iota^{(0)}(e) = \underline{t\cdot E_{n1}}$$

$$\iota^{(0)}(h)(v_\lambda^+) = \lambda(\underbrace{h_0})\cdot v_\lambda^+$$

<u>last time</u> : $h_0 = K + E_{nn} - E_{11}$

$\Rightarrow$ all $k_i \in \mathbb{Z}_{\geq 0}$. <u>Done!</u> ∎

- K - central $\Rightarrow$ K acts on L_λ via the scalar
$$K = h_0 + h_1 + \_\_ + h_{n-1}$$
equal to
$$\lambda(h_0+h_1+\_\_+h_{n-1}) = k_0+k_1+\_\_+k_{n-1}$$

- $\lambda = k_0\tilde{\omega}_0 + k_1\tilde{\omega}_1 + \_\_ + k_{n-1}\tilde{\omega}_{n-1}$

Def: the level of the repr-n.

So: Unitary repr-n have levels $\in \mathbb{Z}_{\geq 0}$

Today (& Thursday) : "Sugawara Construction"

Generalization of $\mathcal{A} \curvearrowright F_\mu \rightsquigarrow \boxed{Vir \ltimes \mathcal{A} \curvearrowright F_\mu.}$

Here: $\mathcal{A}$ can be viewed as an affinization of 1-dim Lie alg.

$\mathfrak{h} = \mathbb{C} \rightsquigarrow \hat{\mathfrak{h}} \simeq \mathcal{A}.$

Setting: $\mathfrak{g}$ - f.dim. Lie alg/$\mathbb{C}$

$(\cdot,\cdot)$ - invariant symm. bilinear on $\mathfrak{g}$ (non-necessarily nondegenerate).

$\rightsquigarrow L\mathfrak{g} = \mathfrak{g}[t, t^{-1}] \rightsquigarrow \boxed{\hat{\mathfrak{g}} \underset{\substack{\text{using} \\ \text{the 2-cocycle} \\ \text{from Week \#1}}}{=} L\mathfrak{g} \oplus \mathbb{C}K.}$

Def: $k \in \mathbb{C}$ is non-critical for $(\mathfrak{g}, (\cdot,\cdot))$ iff

$$\boxed{k \cdot (\cdot,\cdot) + Kil_{\mathfrak{g}} \text{ - non-degenerate}}$$

Recall: $\boxed{Kil_{\mathfrak{g}}(a,b) = Tr_{\mathfrak{g}}(ad(a)ad(b))}$

Def: A $\hat{\mathfrak{g}}$-module M is admissible if $\forall v \in M\ \exists N \gg 0$ s.t. $a t^n (v) = 0\ \ \forall a \in \mathfrak{g},\ \forall n \geq N$.

$\boxed{\text{Lemma 1}}$ $\exists!\ W \xrightarrow{\eta_{\hat{\mathfrak{g}}}} \mathrm{Der}(\hat{\mathfrak{g}})$ – Lie alg. homom. (with alg.)

$$\text{s.t.}\quad W \ni f\partial_t \longmapsto \eta_{\mathfrak{g}}(f\partial_t)\Big(\overbrace{(\underset{L\mathfrak{g}}{g},\ \underset{\mathbb{C}}{\alpha})}^{\in\,\hat{\mathfrak{g}}}\Big) = (f g',\ 0) \in L\mathfrak{g}_{\hat{\mathfrak{g}}}$$

Exercise: Prove Lemma 1!

! $\boxed{\text{Outcome}}$: We have $\underbrace{\boxed{W \ltimes \hat{\mathfrak{g}}} \rightsquigarrow \boxed{\mathrm{Vir} \ltimes \hat{\mathfrak{g}}}}_{\mathrm{Vir} \xrightarrow[c=0]{\text{kill}} W \longrightarrow \mathrm{Der}(\hat{\mathfrak{g}})}$

Theorem 2 (Sugawara Construction):

Let $k \in \mathbb{C}$ be non-critical for $(\mathfrak{g}, (\cdot,\cdot))$,

let M be an admissible $\hat{\mathfrak{g}}$-module of level k.

Then: $\hat{\mathfrak{g}} \curvearrowright M$ extends to $\boxed{Vir \ltimes \hat{\mathfrak{g}} \curvearrowright M}$

Explicitly:
$$\boxed{\underset{\substack{\cup\!\!\!\!\mid \\ Vir}}{L_n} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{a \in B} :a_m a_{n-m}:}$$

with the central charge

$$\boxed{c = k \cdot \sum_{a \in B} (a,a)}$$

Here: * B is a basis of $\mathfrak{g}$ orthonormal w.r.t. $\overbrace{k(\cdot,\cdot) + \frac{1}{2} Kil}^{\text{assumed to be non-deg. (as } k\text{-non-critical)}}$

* $a_n = a \cdot t^n$; $a \in \mathfrak{g}$, $n \in \mathbb{Z}$

* $:a_m a_l: \overset{def}{=} \begin{cases} a_m a_l, & \text{if } m \leq l \\ a_l a_m, & \text{if } m > l. \end{cases}$

Rmk: a) $\forall n \in \mathbb{Z}$, $\forall v \in M$: $L_n(v)$ is well-defined

b) L_n does not depend on the choice of B.

c) For $\mathfrak{g} = \mathbb{C}$ - trivial Lie alg $\rightsquigarrow$ Thm 2 recovers

$(\cdot,\cdot)$ s.t. $(1,1)=1$

our old result $\boxed{Vir \ltimes \mathcal{A} \curvearrowright F_\mu.}$

(for the particular choice $M = F_\mu$)

Proof - next lecture.

We'll show: 1) $[L_n, b_r] = -r\, b_{n+r}$ $\quad \forall r,n \in \mathbb{Z}$, $\forall b \in \mathfrak{g}$

$\Downarrow$

2) $[L_n, L_m] = (n-m) L_{n+m} + \frac{n^3-n}{12} \delta_{n,-m} \cdot k \sum_{a \in B} (a,a)$