Lecture 17

03/25/2021

Last time: Sugawara Construction

Setup:
- $\mathfrak{g}$ – f.d. Lie alg / $\mathbb{C}$
- $(\cdot,\cdot)$ – invariant bilinear form $(\cdot,\cdot)$
- $k \in \mathbb{C}$, s.t. $\underbrace{k(\cdot,\cdot) + \frac{1}{2}\mathrm{Kil}(\cdot,\cdot)}_{=:\langle\cdot,\cdot\rangle}$ – nondeg.
- M – admissible $\hat{\mathfrak{g}}$-module, i.e.
  $\forall v \in M \ \exists N \ \forall x \in \mathfrak{g}, \forall n \geq N \quad xt^n(v) = 0.$

Thm 1 (Sugawara): $\hat{\mathfrak{g}} \curvearrowright M$ can be extended to

$$\mathrm{Vir} \ltimes \hat{\mathfrak{g}} \curvearrowright M \quad \text{via}$$

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{a \in B} :a_m a_{n-m}: \longleftarrow B \text{ – orthogonormal basis of } \mathfrak{g} \text{ w.r.t } \langle\cdot,\cdot\rangle$$

with central charge

$$c = k \cdot \sum_{a \in B} (a,a)$$

Proof (Will take the next 6 pages - be patient! ☺)

Lemma 1 For any $x \in \mathfrak{g}$, we have

$$\boxed{\sum_{a \in B} \left([x,a] \cdot a + a \cdot [x,a] \right) = 0}$$

B - orthonormal basis w.r.t. any (in our case $\langle \cdot , \cdot \rangle$) nondegen. invariant pairing on $\mathfrak{g}$.

Note: $\forall y \in \mathfrak{g}$: $\boxed{y = \sum_{a \in B} (y, a) \cdot a = \sum_{a \in B} a(a|y)}$ (*)

So: $$\sum_{a \in B} a \cdot \underbrace{[x,a]} \overset{(*)}{=} \sum_{\substack{a \in B \\ a' \in B}} a \cdot ([x,a], a') \cdot a' = - \sum_{a, a' \in B} \underbrace{a \cdot (a, [x,a'])} \cdot a' \overset{(*)}{=} - \sum_{a' \in B} [x,a'] \cdot a'$$
$$= - \sum_{a \in B} [x,a] \cdot a.$$

Lemma 2 $$\boxed{\sum_{a \in B} [x,a] \otimes a + a \otimes [x,a] = 0}$$ ← equality in $\mathfrak{g} \otimes \mathfrak{g}$.

Exercise!

Use the same reasoning!

Note: Lemma 2 ⇒ Lemma 1 (use $\mathfrak{g} \otimes \mathfrak{g} \longrightarrow U(\mathfrak{g})$)

Lemma 3 $\forall x \in \mathfrak{g}$: $\boxed{\sum_{a \in B} [a,[a,x]] = \sum_{a \in B} Kil(x,a) \cdot a}$

▸ $Kil(x,y) \overset{\text{by definition}}{=\!=} Tr_{\mathfrak{g}}(ad(x)ad(y)) = Tr_{\mathfrak{g}}\left(\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ z & \longmapsto & [x,[y,z]] \end{array}\right)$

Pick a basis $\{c_1, \ldots, c_m\}$ of $\mathfrak{g}$, dual basis $\{c_1^*, \ldots, c_m^*\}$ of $\mathfrak{g}^*$, so that

$$\boxed{Kil(x,a) = \sum_{j=1}^{m} c_j^*([x,[a,c_j]])}$$

Then:

$$\sum_{a \in B} Kil(x,a) \cdot a = \sum_{\substack{a \in B \\ 1 \le j \le m}} c_j^*([x,[a,c_j]]) \cdot a = \sum_{j=1}^{m} \left(\sum_{a \in B} c_j^*([x,[a,c_j]] \cdot a) \right) =$$

Note (Lemma 2 $\Rightarrow$): $\sum_{a \in B} [a,c_j] \otimes a = -\sum_{a \in B} a \otimes [a,c_j]$. $\quad (\mathfrak{g} \otimes \mathfrak{g} \to U\mathfrak{g})$

$$= -\sum_j \sum_{a \in B} c_j^*([x,a]) \cdot [a,c_j] = -\sum_{a \in B} [a, \underbrace{\sum_{j=1}^{m} c_j^*([x,a]) \cdot c_j}_{=[x,a]}] = -\sum_{a \in B} [a,[x,a]]$$

$$= \sum_{a \in B} [a,[a,x]]$$

Corollary 1 : $\forall x \in \mathfrak{g}$: $\boxed{x = k \sum_{a \in B} (x,a) \cdot a + \frac{1}{2} \sum_{a \in B} [[x,a],a]}$

▸ B-orthonormal basis w.r.t. $k(\cdot,\cdot) + \frac{1}{2} Kil(\cdot,\cdot) = \langle \cdot,\cdot \rangle$

$$x = \sum_{a \in B} \langle x,a \rangle \cdot a = k \sum_a (x,a) \cdot a + \frac{1}{2} \sum_a Kil(x,a) \cdot a$$

Lemma 3.

$\boxed{1^{st} \text{ part}}$ Thm

In the def-n $Vir \ltimes \hat{\mathfrak{g}}$ from last time we have:

$$\boxed{[L_n, b_\tau] = -\tau b_{n+\tau}}$$

($L_n \longleftrightarrow -t^{n+1}\partial_t$

$\boxed{\text{Exercise}}$: Verify the f-la to the left is equivalent to the semidirect product from last time)

$$\boxed{L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{a \in B} :a_m a_{n-m}:}$$

Here, we approximate ∞ sums by finite ones

$$L_n = \frac{1}{2} \lim_{N \to \infty} \sum_{a \in B} \sum_{m : |m - \frac{n}{2}| \le N} :a_m a_{n-m}:$$

$:\ldots:$ is ignored in any commutators

$$[b_\tau, L_n] = \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{m: |m-\frac{n}{2}| \le N} \Big(\underbrace{[b_\tau, a_m]}_{[b,a]_{\tau+m} + \underline{\underline{K}} \cdot \alpha(b_\tau, a_m)} \cdot a_{n-m} + a_m \overbrace{[b_\tau, a_{n-m}]}^{[b,a]_{n-m+\tau} + \underline{\underline{K}} \cdot \alpha(b_\tau, a_{n-m})} \Big) =$$

$\alpha(b_\tau, a_m) = 0$ unless $m = -\tau$

$\alpha(b_\tau, a_{n-m}) = 0$ unless $n-m = -\tau \Leftrightarrow m = n+\tau$

$$= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{m: |m-\frac{n}{2}| \le N} \Big([b,a]_{\tau+m} \cdot a_{n-m} + a_m \cdot [b,a]_{n-m+\tau} + K \cdot \alpha(b_\tau, a_m) + K \cdot \alpha(b_\tau, a_{n-m}) \Big)$$

$$= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{m: |m-\frac{n}{2}| \le N} \Big([b,a]_{\tau+m} \cdot a_{n-m} + a_m \cdot [b,a]_{n-m+\tau} \Big)$$

$$+ \underbrace{\boxed{\frac{1}{2} K \sum_{a \in B} \Big(\tau \cdot (b,a) \cdot a_{n+\tau} + \tau \cdot (b,a) \cdot a_{n+\tau} \Big)}}_{= \tau \cdot K \cdot \sum_{a\in B} (b,a) \cdot a_{n+\tau}}$$

Lemma 2

$=$

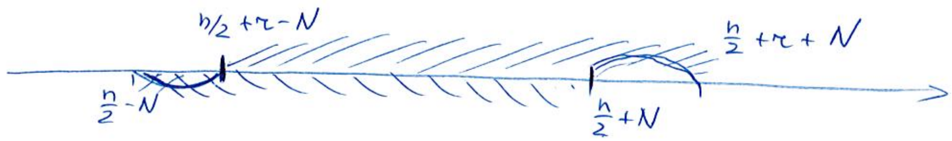
$$\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{m:\,|m-\frac{n}{2}|\le N}\underbrace{\left([b,a]_{\tau+m}\cdot a_{n-m}-[b,a]_m\cdot a_{n-m+\tau}\right)}\quad +$$

$$+\,\tau\cdot k\cdot\sum_{a\in B}(b,a)\cdot a_{n+\tau}$$

$$=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Bigg(\sum_{m:\,|m-\tau-\frac{n}{2}|\le N}-\sum_{m:\,|m-\frac{n}{2}|\le N}\Bigg)[b,a]_m\cdot a_{\tau+n-m}\quad+\tau k\cdot\sum_{a\in B}(b,a)a_{n+\tau}\ \boxed{=}$$

$$-N+\tau+\tfrac{n}{2}\le m\le N+\tau+\tfrac{n}{2}$$

Assume : $\tau>0$



Exercise: Treat the case $\tau\le 0$ in a similar fashion!

acts by 0 on any fixed v for N big enough

$$=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\Bigg(\sum_{\frac{n}{2}+N<m\le\frac{n}{2}+\tau+N}[b,a]_m\cdot a_{\tau+n-m}-\underbrace{\sum_{\frac{n}{2}-N\le m<\frac{n}{2}+\tau-N}[b,a]_m\cdot a_{\tau+n-m}}\Bigg)$$

as $N\gg 1\Rightarrow\tau+n-m\gg 1$.

$\Downarrow$

this sum acts by 0 on any vector (as N becomes very big).

$$+\,\underbrace{\tau\cdot k\cdot\sum_{a\in B}(b,a)\cdot a_{n+\tau}}$$

Make N big enough and act on $v\in M$.

$$=\frac{1}{2}\lim_{N\to\infty}\sum_{a\in B}\sum_{\frac{n}{2}+N<m\le\frac{n}{2}+\tau+N}[[b,a],a]_{n+\tau}+k\cdot\underbrace{d([b,a]_m,a_{\tau+n-m})}_{=0\ \text{b/c}\ ([b,a],a)=(b,\underbrace{[a,a]}_{0})=0}+\tau\cdot k\cdot\sum_{a\in B}(b,a)\cdot a_{n+\tau}$$

$$=\underbrace{\frac{1}{2}\tau\sum_{a\in B}[[b,a],a]_{n+\tau}+\tau\cdot k\cdot\sum_{a\in B}(b,a)\cdot a_{n+\tau}}\ \overset{\text{Corollary 1}}{=}\ \tau\cdot b_{n+\tau}.$$

2nd Part

Theorem

$$\underbrace{[L_n, L_m] - (n-m) L_{n+m}}_{= LHS} \overset{?}{=} \frac{n^3-n}{12}\,\delta_{n,-m}\cdot k \sum_{a\in B}(a,a)$$

- LHS commutes with all b_r $\forall b \in \mathfrak{g}$, $\forall r \in \mathbb{Z}$. (follows from 1st part).
- $[L_0, b_r] \overset{1^{st}\text{ part}}{=\!=} -r\, b_r \implies [L_0, \underline{LHS}] = -(n+m)\cdot(LHS)$

  But L_0 is a sum of quadratic terms in b_r's $\Rightarrow$ commutes with LHS $\Rightarrow$

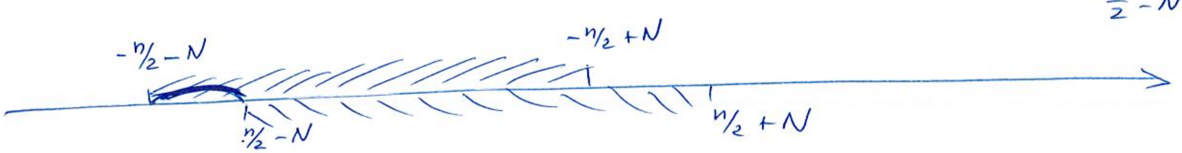
  $\Rightarrow LHS = 0$ unless $n+m=0$.
- Finally, let's treat $m=-n$.

$$[L_n, L_{-n}] = \Big[\frac{1}{2}\lim_{N\to\infty} \sum_{a\in B}\sum_{m:\,|m-\frac{n}{2}|\le N} a_m a_{n-m},\ L_{-n}\Big] \overset{\text{Part 1}}{=\!=}$$

$$= \frac{1}{2}\lim_{N\to\infty}\Big[\sum_{\substack{a\\ m:\,|m-\frac{n}{2}|\le N}} m\cdot a_{m-n}\, a_{n-m} + \sum_{\substack{a\\ m:\,|m-\frac{n}{2}|\le N}} (n-m)\cdot a_m\cdot a_{-m}$$

$$[L_n, L_{-n}] = 2n L_0 =$$

$$= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \Bigg(\sum_{-\frac{n}{2}-N \le m \le -\frac{n}{2}+N} (m+n) \underbrace{a_m a_{-m}} + \sum_{+\frac{n}{2}-N \le m \le \frac{n}{2}+N} (n-m) \underbrace{a_m a_{-m}} - \sum_{\frac{n}{2}-N \le m \le -\frac{n}{2}+N} 2n \cdot \underbrace{:a_m a_{-m}:} \Bigg) =$$



$$= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \Bigg\{ \sum_{-\frac{n}{2}-N \le m < \frac{n}{2}-N} (m+n) \underbrace{a_m a_{-m}}_{\text{kills } v\in M\ \forall N\gg 1} + \sum_{-\frac{n}{2}+N < m \le \frac{n}{2}+N} (n-m) \underbrace{a_{-m} a_m}_{-\|-}$$

$$+ k(a,a) \Big(\sum_{1\le m\le -\frac{n}{2}+N} m(m+n) + \sum_{1\le m\le \frac{n}{2}+N} m(n-m) \Big) \Bigg\}$$

$$\frac{1}{2} k \sum_{a\in B} (a,a) \Big(\ \parallel \ \frac{n^3-n}{6} \Big) = k \frac{n^3-n}{12} \sum_{a\in B} (a,a)$$

This completes our proof of the Sugawara Theorem! ✓

$\boxed{\text{Application 1}}$: $\mathfrak{g}$ – abelian

$$\left.\begin{array}{l} \mathrm{Kil} \equiv 0 \\ (\cdot,\cdot) - \text{non-degen.},\ k \neq 0. \end{array}\right\} \Rightarrow \langle\cdot,\cdot\rangle = k\cdot(\cdot,\cdot)$$

So: $\boxed{c = k \sum_{a\in B} (a,a) = \dim(\mathfrak{g})}$

In the simplest case: $\left.\begin{array}{l} \mathfrak{g} = \mathbb{C} \\ M = F_\mu \text{ – Fock mod} \end{array}\right\} \Rightarrow$ get level 1 repr. of $Vir \curvearrowright F_\mu$.

Our old construction.

$\boxed{\text{Application 2}}$: $\mathfrak{g}$ – simple f.d. Lie alg.

Note : Any invariant pairing is a multiple of Kil $(\cdot,\cdot)$

Standard choice of $(\cdot,\cdot)$ is such that it induces a bil. form on $\mathfrak{g}^*$ with $\boxed{(\alpha,\alpha)=2 \text{ for long roots}}$

Equivalently: $\boxed{(\theta,\theta)=2, \text{ where } \theta = \text{max/highest root of } \mathfrak{g}}$

Def : The $\boxed{\text{dual Coxeter number } h^\vee}$ of $\mathfrak{g}$ is defined via:

$$\boxed{h^\vee = 1 + (\theta,\rho)} \qquad \rho = \frac{1}{2}\underbrace{\sum_{\alpha\in\Delta^+}}_{\text{positive roots}}\alpha = \sum_i \underbrace{\omega_i}_{\text{fund. weights}}$$

Table :

Type	$h^\vee$
A_n (sl_{n+1})	$n+1$
B_n (so_{2n+1})	$2n-1$
C_n (sp_{2n})	$n+1$
D_n (so_{2n})	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

$\boxed{\text{Prop 1}}$: $\boxed{Kil(a,b) = 2h^\vee (a,b)}$ ◀— As any two inv. pairings are proportional to each other ($\mathfrak{g}$-simple f.d.) it's really about computing this constant $2h^\vee$

▶ Recall Casimir elt $\overset{\in\, U(\mathfrak{g})}{C} = \sum_{a\in B'} a^2$, B'- orthonormal basis w.r.t. $(\cdot,\cdot)$.

It is central by Lemma 1 $\Rightarrow$ acts by constant γ_λ on Verma modules M_λ

$\downarrow$

irred. h.wt.mod. L_λ

Hwk Problem : The above constant γ_λ is given by:

$$\boxed{\gamma_\lambda = (\lambda, \lambda + 2\rho)}$$

$[e_\alpha, e_\theta] = 0$ b/c θ- maximal
$[h_\alpha, e_\theta] = \alpha(\theta)\cdot e_\theta$
adjoint repr-n has h.wt $= \theta$

Let's look at the adjoint repr. $\mathfrak{g} \curvearrowright \mathfrak{g}$ $\quad x \mapsto ad(x)$

$$\underset{\|}{Tr_{\mathfrak{g}}(C)} = \gamma_\theta \cdot \dim\mathfrak{g} = (\theta, \theta+2\rho)\cdot \dim\mathfrak{g} = \underbrace{(\theta,\theta+2\rho)}_{=\gamma_\theta}\cdot \sum_{a\in B'} (a,a)$$

$$Tr_{\mathfrak{g}}\Big(\sum_{a\in B'} ad(a)^2\Big) = \sum_{a\in B'} Kil(a,a)$$

$\Downarrow$

So: $Kil(a,b) = \gamma_\theta \cdot (a,b)$

But: $\gamma_\theta = (\theta, \theta+2\rho) = 2 + (2\rho, \theta) = 2h^\vee$

Corollary For simple f.d. Lie alg. $\mathfrak{g}$, k - non-critical iff

$$\boxed{k \neq -h^\vee}$$

$$\boxed{k = -h^\vee \longleftarrow \text{"critical level"}}$$

Restate Thm 1 in flat setup (this formulation is more commonly known):

Thm 2 (Sugawara Constr): $\mathfrak{g}$ - simple f.d. Lie alg with standard pairing

$\hat{\mathfrak{g}} \curvearrowright M$ - admissible module of non-critical level $k_{\neq -h^\vee}$ $\rightsquigarrow$ $Vir \curvearrowright M$

! $$L_n = \frac{1}{2(k+h^\vee)} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}:$$

B' ← orthonormal basis w.r.t. $(\cdot,\cdot)$ - standard choice

with central charge

! $$c = \frac{k}{k+h^\vee} \sum_{a\in B'} \underbrace{(a,a)}_{=1} = \frac{k \dim \mathfrak{g}}{k+h^\vee}$$

Corollary Any $\hat{\mathfrak{g}}$-module M (non-critical level) realized as a quot-t of a Verma module M_λ^+ (Exercise: $\Rightarrow M$- admissible) can be naturally endowed with a $\tilde{\mathfrak{g}}$-mod str.

Recall: $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$ ← "degree operators"

$[d, k] = 0$

$[d, xt^k] = k \cdot xt^k$.

Know: $[L_0, b_r] = -r\, b_r \implies \underbrace{d = -L_0}$ ← i.e. $-L_0$ provides the degree operator ∎

Prop 2 (Hwk Problem): $\underbrace{\hat{\mathfrak{g}} \curvearrowright M - \text{unitary}} \implies Vir \ltimes \hat{\mathfrak{g}} \curvearrowright M$- unitary.

$\Downarrow$

$k \in \mathbb{Z}_{\geq 0}$ ($k \in \mathbb{Z}_{>0}$ unless M is trivial)

$\Downarrow$ Exercise (look up at the table)

$$\frac{k \dim \mathfrak{g}}{k + h^\vee} > 1$$

$$\boxed{c = \frac{k \dim \mathfrak{g}}{k + h^\vee}}$$

For $\mathfrak{g} = \mathfrak{sl}_n$ we proved this last time

But we already know unitarity at $\begin{cases} c > 1 \\ h \geq 0 \end{cases}$

! We will update this next time to provide unitary repr-s with $0 \leq c \leq 1$

Q What happens when $k = -h^\vee$?

$$\boxed{T_n :\overset{def}{=} \frac{1}{2} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}:}$$

Take the f-las for L_n and multiply by $(k+h^\vee)$ in denominator

Following Part 1 of the proof of Thm 1:

Corollary $[T_n, a_m] = 0 \quad \forall n,m\in\mathbb{Z} \ \forall a \in \mathfrak{g}$.

$$\Downarrow$$

$\{T_n\}_{n\in\mathbb{Z}}$ – central el-s in a certain completion $U(\hat{\mathfrak{g}})^\wedge$

This basically follows from the last line in our proof of 1st part of Theorem ← Check this out!

That's a fundamental feature of the critical level!