Lecture 18

03/30/2021

Last time:

- Proved Sugawara Construction

(1) $[b_r, L_n] = r\, b_{n+r}$

(2) $[L_n, L_m] - (n-m) L_{n+m} = \frac{n^3-n}{12} \delta_{n,-m} \cdot k \cdot \sum_{a \in B} (a,a)$

Here: $b_r = b \otimes t^r \in \hat{\mathfrak{g}}$

L_n – Virasoro's guy $= \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{a \in B} :a_m a_{n-m}:$

Casimir element

$\mathfrak{g}$ (f.d. Lie alg.), $(\cdot,\cdot)$ - nondeg. inv. pairing on $\mathfrak{g}$

$$\mathrm{Cas} = \Omega_0 = \sum a_i a^i = \sum_{a \in B'} a \cdot a \in U(\mathfrak{g})$$

Casimir element

B' ← orthonormal basis

$\{a_i\}$, $\{a^i\}$ – dual w.r.t $(\ ,\ )$ bases of $\mathfrak{g}$

$\boxed{[\Omega_0, \mathfrak{g}] = 0} \Rightarrow \boxed{\Omega_0 \text{ – central}}$

Casimir tensor: $\Omega = \sum a_i \otimes a^i \in U\mathfrak{g} \otimes U\mathfrak{g}$.

Lemma: $[\Omega, x \otimes 1 + 1 \otimes x] = 0 \quad \forall x \in \mathfrak{g}$

• Key Application: $\mathfrak{g}$ - simple f.d.

Canonical inv. form $(\cdot,\cdot)$ s.t. $(\theta,\theta)=2$, θ = highest root.

$\parallel$ Def: $h^\vee = 1+(\theta,\rho)$ ← dual Coxeter number

Last time: $\boxed{Kil(a,b) = 2h^\vee \cdot (a,b) \quad \forall a,b \in \mathfrak{g}}$

Thm (Sugawara for simple f.d.): $k \neq -h^\vee$

M – admissible level k $\mathfrak{g}$-module

⇝ Get $Vir \curvearrowright M$

via $L_n = \frac{1}{2(k+h^\vee)} \sum_{m\in\mathbb{Z}} \sum_{a\in B'} :a_m a_{n-m}:$

↖ orthonormal basis w.r.t. $(\cdot,\cdot)$

with central charge

$$c = \frac{k \cdot \dim \mathfrak{g}}{k+h^\vee}$$

- $\boxed{\text{Critical level}}$: $k = -h^\vee$.

Key Property:

Define : $\boxed{T_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{a \in B'} :a_m a_{n-m}:}$ $\longleftarrow$ central el-s of $U(\hat{\mathfrak{g}})^\wedge$ $\swarrow$ completion

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- $\hat{\mathfrak{g}} \curvearrowright M$ - unitary $\xRightarrow{\text{Hwk Problem}}$ Vir $\curvearrowright M$ - unitary.

$\Downarrow$

$k \in \mathbb{Z}_{\geq 0}$ ($k \in \mathbb{Z}_{>0}$ if M is nontrivial) $\Longrightarrow c = \frac{k \dim \mathfrak{g}}{k + h^\vee} > 1$

$\Rightarrow$ Don't get anything new.

$\boxed{Q}$: Can we update the above construction to get Vir - unitary repr. with $0 < c < 1$?

$\boxed{A}$: "Coset Construction".

Input: • $\mathfrak{g} \supseteq \mathfrak{p}$ – two fin. dim. Lie alg. $\rightsquigarrow$ $L\mathfrak{g} \supseteq L\mathfrak{p} \rightsquigarrow \hat{\mathfrak{g}} \supseteq \hat{\mathfrak{p}}$.

• $(\cdot,\cdot)$ – inv. on $\mathfrak{g} \Rightarrow$ also on $\mathfrak{p}$

• $k \in \mathbb{C}$ – non-critical for both $\mathfrak{g}, \mathfrak{p}$.

• M – admissible $\hat{\mathfrak{g}}$-mod of level $k \Rightarrow$ also admiss. $\hat{\mathfrak{p}}$-mod of level k.

Using Sugaware construction $\Rightarrow$ $\{L_n^{\mathfrak{g}}\}_{n\in\mathbb{Z}}$, $\{L_n^{\mathfrak{p}}\}_{n\in\mathbb{Z}}$ – two different Vir-action on M.

$c_{\mathfrak{g}}$, $c_{\mathfrak{p}}$ central charges

Thm (Goddard – Kent – Olive '85): Set $L_i = L_i^{\mathfrak{g}} - L_i^{\mathfrak{p}}$ $\forall i \in \mathbb{Z}$; $c := c_{\mathfrak{g}} - c_{\mathfrak{p}}$. Then:

(a) $\{L_i\}$ define a Vir-action on M with central charge c

(b) $[L_n, L_m^{\mathfrak{p}}] = 0$ $\forall n, m$.

Proof

- $\forall b \in \mathfrak{h}\ \forall r \in \mathbb{Z}$: $\left.\begin{array}{l} [L_n^{\mathfrak{g}}, b_r] = -r\, b_{n+r} \\ [L_n^{\mathfrak{h}}, b_r] = -r\, b_{n+r} \end{array}\right\} \Rightarrow [L_n, b_r] = 0$

$\Rightarrow \{L_n\}_{n\in\mathbb{Z}}$ commute with $\hat{\mathfrak{h}}$ $\Rightarrow$ $[L_n, L_m^{\mathfrak{h}}] = 0$ $\forall n, m$ $\Rightarrow$ part (b).

- $[L_n, L_m] = [L_n, L_m^{\mathfrak{g}} - L_m^{\mathfrak{h}}] \overset{(b)}{=} [L_n, L_m^{\mathfrak{g}}] = [L_n^{\mathfrak{g}} - L_n^{\mathfrak{h}}, L_m^{\mathfrak{g}}] =$

$[L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{h}}, \underbrace{L_m^{\mathfrak{g}}}_{L_m + L_m^{\mathfrak{h}}}] \overset{(b)}{=} \underbrace{[L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}]}_{(n-m)L_{n+m}^{\mathfrak{g}} + \frac{n^3-n}{12}\delta_{n,-m} c_{\mathfrak{g}}} - \underbrace{[L_n^{\mathfrak{h}}, L_m^{\mathfrak{h}}]}_{(n-m)L_{n+m}^{\mathfrak{h}} + \frac{n^3-n}{12}\delta_{n,-m} c_{\mathfrak{h}}} =$

$= (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m} \cdot c$.

Let's apply this to a very particular case!

$\mathfrak{a}$ - simple f.d. Lie alg

$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a} \xleftarrow{\text{diagonally}} \mathfrak{a}$

$(x, x) \longleftarrow x$

$\boxed{\mathfrak{p} \subseteq \mathfrak{g}} \longleftarrow$ Apply previous result to this pair.

$\mathfrak{p}$ = diagonal copy of $\mathfrak{a}$

- $(\cdot, \cdot)$ - canonical form on $\mathfrak{a} \Rightarrow$ inv. form on $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$
- V', V'' - two admissible $\hat{\mathfrak{a}}$-mod of levels k', k'', $k' \neq -h^\vee_{\mathfrak{a}} \neq k''$.

$\boxed{\text{Sugawara}} \Rightarrow \{L_i'\}, \{L_i''\}$ - Sugawara construction of Vir action with central charges $\dfrac{k' \dim \mathfrak{a}}{k' + h^\vee_{\mathfrak{a}}}$, $\dfrac{k'' \dim \mathfrak{a}}{k'' + h^\vee_{\mathfrak{a}}}$

$\{L_i'\} \circlearrowright V'$, $\{L_i''\} \circlearrowright V''$

$\underline{V := V' \otimes V''} \circlearrowleft \hat{\mathfrak{g}} \supseteq \hat{\mathfrak{p}} \simeq \hat{\mathfrak{a}}$

As a $\hat{\mathfrak{g}}$-module V has level $k' + k''$

$\{L_i^{\mathfrak{g}}\}$ define Vir $\curvearrowright V = V_1 \otimes V_2$

$\parallel$

$\boxed{L_i' \otimes 1 + 1 \otimes L_i''}$

of central charge $= \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee}\right) \dim \mathfrak{g}$

$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$

$L_n^{\mathfrak{g}} = (\ldots) \sum_{m \in \mathbb{Z}} \sum_{a \in B\text{- orthonormal basis of } \mathfrak{a}\oplus\mathfrak{a}} :a_m a_{n-m}:$

$\hat{\mathfrak{a}} \simeq \hat{\mathfrak{g}}^{\sigma}$ $\curvearrowright V$ $\rightsquigarrow$ $\{L_i^{\sigma}\}$ define Vir $\curvearrowright V$

$\bigcap$

$\hat{\mathfrak{g}}$

Sugawara

of central charge $= \frac{k'+k''}{k'+k''+\mathfrak{a}} \dim \mathfrak{a}$

Recall:

$\mathfrak{h}$ - Lie alg $\curvearrowright V_1, V_2 \implies \mathfrak{h} \curvearrowright V_1 \otimes V_2$ via

$x \longmapsto x \otimes 1 + 1 \otimes x$

$\Downarrow$

$x_1 x_2 \longmapsto \underline{x_1 x_2 \otimes 1} + 1 \otimes \underline{x_1 x_2} + (x_1 \otimes x_2 + x_2 \otimes x_1)$

G-K-O thm tells: $\qquad\qquad\qquad V', V'' \curvearrowleft \hat{\mathfrak{g}}$

Prop 1: Get Vir $\curvearrowright \overbrace{V' \otimes V''}^{V}$ given by

$$L_n = \left(\frac{1}{2(k'+h^\vee)} - \frac{1}{2(k'+k''+h^\vee)} \right) \sum_{m \in \mathbb{Z}} \sum_{a \in B} :a_m a_{n-m}: \otimes 1 \; +$$

$$+ \left(\frac{1}{2(k''+h^\vee)} - \frac{1}{2(k'+k''+h^\vee)} \right) \sum_{m \in \mathbb{Z}} \sum_{a \in B} 1 \otimes :a_m a_{n-m}: \; -$$

$$- \frac{1}{k'+k''+h^\vee} \sum_{m \in \mathbb{Z}} \sum_{a \in B} a_m \otimes a_{n-m}$$

with central charge

$$c = \left(\frac{k'}{k'+h^\vee} + \frac{k''}{k''+h^\vee} - \frac{k'+k''}{k'+k'+h^\vee} \right) \dim \mathfrak{g}$$

Moreover, L_n commute with $\hat{\mathfrak{p}}$-action.

Example : $\mathfrak{g} = \mathfrak{sl}_2 \Rightarrow h^\vee = 2$ and $\dim(\mathfrak{sl}_2) = 3$

Take : $k' = 1$, $k'' = m$

Then:

$$\boxed{c = 3\left(\frac{1}{3} + \frac{m}{m+2} - \frac{m+1}{m+3}\right) = 1 - \frac{6}{(m+2)(m+3)}}$$

Values of c in the "discrete series".

To further elaborate we'll need to develop repr. theory of affine alg-s first.

$\boxed{\text{Today}}$: Simple $\overset{\text{f.d.}}{\vee}$ Lie alg-s $\longleftarrow$ We will just recall the very basics of simple f.d. Lie algebras to motivate our upcoming study of contragredient Lie algebras

- $\mathfrak{h} \subseteq \mathfrak{g}$ – Cartan subalgebra

  i.e. a maximal commut. Lie subalg of semisimple el-s

  [Note: not unique, but all are conjugate (by corresponding Lie group G)]

- $\dim \mathfrak{h} =: \mathrm{rk}\, \mathfrak{g}$

- $Kil(\cdot,\cdot)|_{\mathfrak{h}}$ – non-deg (we shall elaborate on that in a few pages)

- $\forall \alpha \in \mathfrak{h}^* \rightsquigarrow \boxed{\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)\cdot x \quad \forall h \in \mathfrak{h}\}}$.

$\boxed{\text{Claim}}$:

- $\mathfrak{g}_0 = \mathfrak{h}$
- $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ – finite is called the root system of $\mathfrak{g}$.
- $\boxed{\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha}$.
- $\forall \alpha \in \Delta$, $\mathfrak{g}_\alpha$ is 1-dim, $\boxed{\mathfrak{g}_\alpha = \mathbb{C} e_\alpha}$ "root vectors"

- Fix $\bar{h} \in \mathfrak{h}$ s.t. $\alpha(\bar{h}) \in \mathbb{R} \setminus \{0\}$ $\forall \alpha \in \Delta$

Exercise: $\bar{h}$ exists.

$$\boxed{\Delta = \Delta_+ \amalg \Delta_-}$$

Δ_+: positive, Δ_-: negative

$\Delta_- = -\Delta_+$.

- $$\boxed{\mathfrak{g} \overset{\text{v. space}}{=} \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+}.$$

- $\mathfrak{h}^*$ has a basis of positive simple roots $\{\alpha_1, \alpha_2, \dots, \alpha_{n = \mathrm{rk}(\mathfrak{g}) = \dim \mathfrak{h}}\} \subseteq \Delta_+$ s.t.

  any $\alpha \in \Delta_+$ can be written as $\alpha = \sum_{i=1}^{n} k_i \alpha_i$, $k_i \in \mathbb{Z}_{\geq 0}$.

Claim:
- If $\alpha + \beta \notin \Delta \amalg \{0\} \Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.
- If $\alpha + \beta \in \Delta \Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
- If $\alpha + \beta = 0 \Rightarrow \underbrace{[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]}_{\subseteq \mathfrak{h} = \mathfrak{g}_0} \neq 0$

$$\boxed{\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \underbrace{\mathbb{C} e_\alpha}_{\mathfrak{g}_\alpha} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_+} \mathbb{C} e_\alpha \oplus \bigoplus_{\alpha\in\Delta_+} \mathbb{C} f_\alpha.}$$

- $f_\alpha \overset{\alpha\in\Delta_+}{=} e_{-\alpha}$ $\qquad [e_\alpha, f_\alpha] = h_\alpha \neq 0.$ $\quad h_\alpha \in \mathfrak{h}$.

- $\alpha = \alpha_i \rightsquigarrow \underline{\underline{e_{\alpha_i}}} = e_i$, $\underline{\underline{f_{\alpha_i}}} = f_i$, $\quad h_i = [e_i, f_i] \in \mathfrak{h}$

Can renormalize e_i, f_i so that $\{e_i, h_i, f_i\}$ is an $\mathfrak{sl}_2$ - triple:

$$\boxed{[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad [e_i, f_i] = h_i}$$

Prop 2. (1) $\{h_1, \ldots, h_n\}$ - form a basis $\mathfrak{h}$

(2) $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i=1}^n$

(3) The following rel-s hold, (these are not defining rel-s!)

$$(*) \quad \boxed{[h_i, h_j] = 0, \quad [h_i, e_j] = \alpha_j(h_i) e_j, \quad [h_i, f_j] = -\alpha_j(h_i) f_j, \quad [e_i, f_j] = \delta_{ij} h_i}$$

! Serre rel-s are not mentioned here!

def Consider a Lie alg. $\underline{\underline{\tilde{\mathfrak{g}}}}$ gen-d by $\{e_i, f_i, h_i\}_{i=1}^{n}$ subject to rel-s $\underline{\underline{(*)}}$.

$$\dim(\tilde{\mathfrak{g}}) \underset{\uparrow}{=} \infty \rightsquigarrow \boxed{\tilde{\mathfrak{g}} \xrightarrow[\text{homom.}]{\text{surj. Lie alg.}} \mathfrak{g}}$$ – with nontrivial (unless $\mathrm{rk}\,\mathfrak{g}=1$). kernel.

unless $\mathrm{rk}\,\mathfrak{g}=1$

- $(\cdot,\cdot)$ – inv. form on $\mathfrak{g}$ as last time, so that $\mathrm{Kil}(\cdot,\cdot) = 2h^\vee \cdot (\cdot,\cdot)$.

$$(\cdot,\cdot)|_{\mathfrak{h}} \text{ – nondeg} \rightsquigarrow \boxed{\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*} \rightsquigarrow \text{nondeg. pairing on } \mathfrak{h}^*.$$

$$h_i \underset{\uparrow}{\longmapsto} \alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$$

$\boxed{\text{Exercise}}$

- $\boxed{a_{ij} := \alpha_j(h_i) = \dfrac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}} \Longrightarrow \boxed{\begin{array}{l}[h_i, e_j] = a_{ij}\cdot e_j \\ [h_i, f_j] = -a_{ij}\cdot f_j\end{array}}$

<u>Prop 3</u> : $A := (a_{ij})_{i,j=1}^{n}$ — Cartan matrix.

The following properties hold:

1) $a_{ii} = 2$

2) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ <u>and</u> $a_{ij} \in \mathbb{Z}_{\leq 0}$ $\forall i \neq j$.

3) A - <u>indecomposable</u>, i.e. $\nexists$ conjugation by permutation of the form $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$

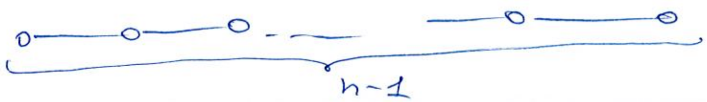
4) A - <u>positive</u>, i.e. $\exists D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, $d_i \in \mathbb{R}_{>0}$

s.t. $D \cdot A$ - symmetric & positive definite.

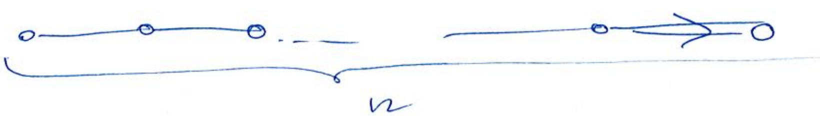
__Thm 2:__ 1) Matrix $A=(a_{ij})$ satisfies the properties 1)–4) above

__iff__ A is a Cartan matrix of simple f.d. Lie alg.

2) A complete classification is via Dynkin diagrams:

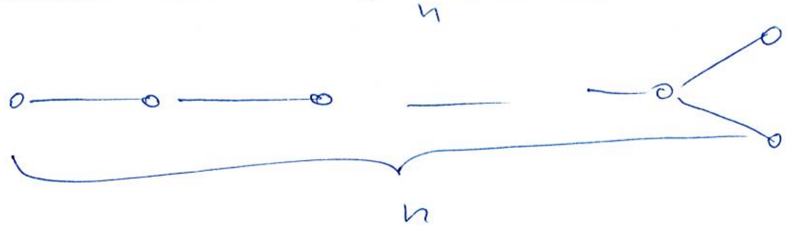


$A_{n-1} = sl_n$, $n \geq 2$



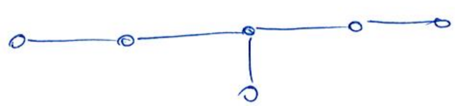
$B_n = so_{2n+1}$, $n \geq 2$

$C_n = sp_{2n}$, $n \geq 3$

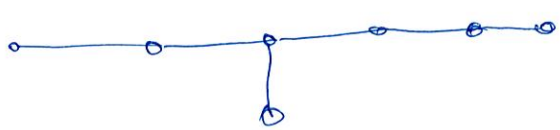


$D_n = so_{2n}$, $n \geq 4$

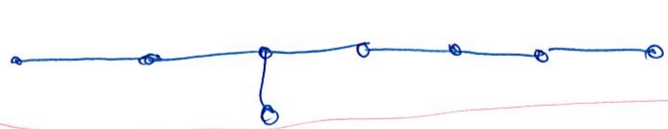
Classical



E_6



E_7



E_8

F_4

G_2

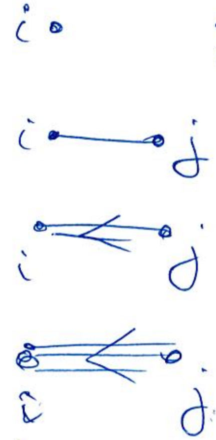
exceptional

$a_{ij} = 0 \iff$ no edges b/w

$a_{ij} = -1 \iff$

$a_{ij} = -2 \iff$

$a_{ij} = -3 \iff$



! direction is always towards the shorter roots

! $a_{ij} \geq -3$ b/c $\begin{pmatrix} a_{ii}=2 & a_{ij} \\ a_{ji} & a_{jj}=2 \end{pmatrix}$ — positive def. $\Rightarrow a_{ij} a_{ji} < 4$.

Thm 3: (1) Let $i \neq j$, then the following Serre rel-s hold:

$$\boxed{\underbrace{\mathrm{ad}(e_i)^{1-a_{ij}}(e_j)}= 0 \qquad \mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0}$$

$$[\underbrace{e_i \ldots [e_i}_{1-a_{ij}}, [e_i, e_j] \ldots] = 0 \qquad \text{— // —}$$

(2) The kernel $(\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g})$ is gen-d by

$$\boxed{\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) \qquad \forall i \neq j}$$

Proof of (1)

$\mathfrak{g} \curvearrowright \mathfrak{g}$ – adjoint $\qquad x \longmapsto \mathrm{ad}\, x\, (y \longmapsto [x,y])$

Consider $f_j \in \mathfrak{g}$

$$\mathfrak{g} \supset \mathfrak{sl}_2^{(i)} = \langle e_i, h_i, f_i \rangle \quad i \neq j.$$

$[e_i, f_j] = 0$ (as $\alpha_i - \alpha_j \notin \Delta \sqcup \{0\}$), $\quad [h_i, f_j] = \overbrace{-a_{ij}}^{\in \mathbb{Z}_{\geq 0}} \cdot f_j \xRightarrow{\mathfrak{sl}_2\text{-theory}} \mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$

otherwise we would see a copy of the Verma module in $\mathfrak{g}$, which contradicts $\dim(\mathfrak{g}) < \infty$ ∎