04/01/2021

LECTURE #19 "Contragredient Lie algebras"

A – $n\times n$ $\mathbb{C}$-valued matrix ← $\boxed{\text{Input}}$

Let Q ("root lattice") be a free abelian gp of rank n with basis $\{\alpha_i\}_{i=1}^n$

$$\boxed{Q = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i}$$

KEY DEFINITION

Def 1: A contragredient Lie alg. corresponding to A is a Q-graded $\mathbb{C}$-Lie alg $\boxed{\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha}$

$\mathfrak{g} = \mathfrak{g}(A)$ generated by $\{e_i, h_i, f_i\}_{i=1}^n$ satisfying:

(1) $[h_i, h_j]=0$, $[h_i, e_j] = a_{ij}\cdot e_j$, $[h_i, f_j] = -a_{ij} f_j$, $[e_i, f_j] = \delta_{ij}\cdot h_i$.

(2) $\mathfrak{g}_0$ has a basis $\{h_i\}_{i=1}^n$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$

(3) Any nonzero Q-graded ideal of $\mathfrak{g}$ has a nonzero intersection with $\mathfrak{g}_0 =: \mathfrak{h}$.

$\boxed{\text{Cor 1}}$: Simple f.d. Lie alg-s are contragredient!

$\boxed{\text{Thm 1}}$: $\forall A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ There exists a unique, up to isom., contragredient Lie alg $\mathfrak{g}$ corresponding to A.

$\Downarrow$

notation $\mathfrak{g}(A)$ is unambigious.

Proof of Thm 1

- As in the previous class, let $\tilde{\mathfrak{g}}(A)$ be the Lie alg. gen-d by $\{e_i, h_i, f_i\}_{i=1}^n$ subject to rel-s (1).

$\boxed{\text{Claim} = \text{Thm 2}}$ $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ & $\tilde{\mathfrak{h}}$ has a basis $\{h_i\}_{i=1}^n$

Lie subalg. gen-d by f_i's; generated by h_i's; gen-d by e_i's

$\tilde{\mathfrak{n}}_+$ is the free Lie alg. in $\{e_i\}_{i=1}^n$

$\tilde{\mathfrak{n}}_-$ —//— $\{f_i\}_{i=1}^n$.

$\boxed{\text{Claim} \Rightarrow \text{Thm 1}}$

- $A \rightsquigarrow \underbrace{\tilde{\mathfrak{g}}(A)}$ ← satisfies (1) + (2).

due to "Q-graded" condition.

does not intersect $\mathfrak{h}$

Let I be the sum of all Q-graded ideals of $\tilde{\mathfrak{g}}(A)$ which do not intersect $\mathfrak{h}$

$\rightsquigarrow \boxed{\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/I}$ (*)

$\mathfrak{g}(A)$ – satisfies (1) ✓

—//— (2) $\leftarrow \underbrace{\text{if } e_i \in I \Rightarrow [f_i, e_i] \in I \Rightarrow \text{\large ↯}}_{\Downarrow\ \mathfrak{g}_{\alpha_i} = \mathbb{C}e_i}$ ✓

$[f_i, e_i] = -h_i$

—//— (3) by construction

$\boxed{\text{Conclusion}}$: $\underset{\text{of } (*)}{\mathfrak{g}(A)}$ – contragredient Lie alg. assoc. to A.

Uniqueness

Assume $\mathfrak{g}'$ - another contragr. Lie alg. associated to A.

$$(1) \Rightarrow \begin{tikzcd} \tilde{\mathfrak{g}}(A) \arrow[r, two heads] & \mathfrak{g}' \\ I \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & 0 \end{tikzcd} \Longrightarrow \boxed{\underset{\mathfrak{g}(A)}{\tilde{\mathfrak{g}}(A)/I \,\|} \twoheadrightarrow \mathfrak{g}'}$$

due to condition (3) for $\mathfrak{g}'$

BUT: if it was not injective, its kernel J would be a Q-graded ideal which must (by (3)) intersect $\mathfrak{h}_{\mathfrak{g}(A)} \Rightarrow \dim \mathfrak{h}_{\mathfrak{g}'} < \dim \mathfrak{h}_{\mathfrak{g}(A)} \Rightarrow$ ↯.

use condition (2) of Def 1.

$$\Rightarrow \boxed{\mathfrak{g}(A) \xrightarrow{\sim} \mathfrak{g}'}$$

Rmk: Given a set X, the *free Lie alg.* on X is a unique (up to isom) Lie alg. $\mathrm{FreeLie}(X)$ together with a set map $X \longrightarrow \mathrm{FreeLie}(X)$, s.t. $\forall$ Lie alg. L $\forall$ set map $X \longrightarrow L$ $\exists!$ Lie alg. hom.

$$\begin{tikzcd} \mathrm{FreeLie}(X) \arrow[r, "\exists!"] & L \\ X \arrow[u] \arrow[ur] & \end{tikzcd}$$

universal enveloping

Exercise: $\mathcal{U}(\mathrm{FreeLie}X) = $ Free Assoc. Alg on the set X.

Proof of Thm 2

$Q_-^{\setminus 0}$-graded part, 0-graded part, $Q_+^{\setminus 0}$-graded part $\; \| \; \bigoplus \mathbb{Z}_{\geq 0}\alpha_i$

- First, we claim that $\tilde{n}_- \oplus \mathfrak{h} \oplus \tilde{n}_+$ is indeed direct!

$$\tilde{\mathfrak{g}}(A) = \langle e_i, h_i, f_i \rangle_{i=1}^n \Big/ \left(\begin{array}{l} [h_i, h_j] = 0 \\ [h_i, e_j] = a_{ij} e_j \\ [h_i, f_j] = -a_{ij} f_j \\ [e_i, f_j] = \delta_{ij} h_i \end{array} \right) \longleftarrow \text{Q-graded with}$$

$\deg(e_i) = \alpha_i = -\deg(f_i)$

$\deg(h_i) = 0$

(as all defining rel-s are homogeneous)

- Actually: $\boxed{\underset{\ni \{f_i\}_{i=1}^n}{\tilde{n}_-} \oplus \underset{\ni \{h_i\}}{\mathfrak{h}} \oplus \underset{\ni \{e_i\}}{\tilde{n}_+} = \tilde{\mathfrak{g}}(A)}$ $\longleftarrow$ The inclusion "$\subseteq$" is obvious!

For "$\supseteq$" it suffices to show: LHS is stable under adjoint action of e_i, h_i, f_i.

Easy Exercise → Obvious
$$\left(\begin{array}{l} \text{e.g. } [h_i, [e_k, e_l]] = a_{ik} \cdot [e_k, e_l] + a_{il} [e_k, e_l] \\ [f_i, [e_k, e_l]] = [[f_i, e_k], e_l] + [e_k, [f_i, e_l]] = -\delta_{ik} \cdot a_{il} \cdot e_l + \delta_{il} \cdot a_{ik} e_k \end{array} \right)$$

Hwk Problem : (a) Construct $\tilde{\mathfrak{g}}(A) \curvearrowright U(\mathfrak{h} \ltimes \text{FreeLie}\{e_i\}_{i=1}^n)$

v.space w/ basis $\{h_i\}_{i=1}^n$, $[h_i, e_j] = a_{ij} e_j$.

Note : this is "universal free Verma module" for $\tilde{\mathfrak{g}}(A)$.

(b) Deduce $\tilde{\mathfrak{n}}_+ \simeq \text{FreeLie}(\{e_i\}_{i=1}^n)$, $\tilde{\mathfrak{h}} \simeq \mathfrak{h}$.

(c) — // — $\tilde{\mathfrak{n}}_-$ . . .

! This concludes the proof of Thm 2. ■

So: $\boxed{A \in \text{Mat}_{n\times n}(\mathbb{C}) \rightsquigarrow \mathfrak{g}(A) \text{ - contragredient Lie alg.}}$

Lemma 1 : (a) If $A' = \sigma A \sigma^{-1}$, σ - permutation matrix $\Rightarrow$ $\boxed{\mathfrak{g}(A') \simeq \mathfrak{g}(A)}$

Hwk Problem (b) If $A' = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$ $\Rightarrow$ $\boxed{\mathfrak{g}(A') \simeq \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)}$

Def 2: $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is called a generalized Cartan matrix if:

(1) $a_{ii} = 2 \quad \forall 1 \le i \le n$.

(2) $\forall i \ne j$: $a_{ij} \in \mathbb{Z}_{\le 0}$ & $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

(3) A-symmetrizable, more precisely, $\exists$ diagonal $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, $d_i \in \mathbb{R}_{>0}$, s.t. $(DA)^T = DA$

$\boxed{\text{Rmk}}$: A - usual Cartan $\Leftrightarrow \begin{cases} A\text{-generalized Cartan AND} \\ DA\text{- positive definite} \end{cases}$

$\boxed{\text{Example } (n=2)}$ $A = \begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix}$, $m \in \mathbb{Z}_{>0}$, is a generalized Cartan matrix (take $D = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$)

For $m=1$: $\mathfrak{g}(A) \simeq \mathfrak{sl}_3$

$m=2$: $\mathfrak{g}(A) \simeq \mathfrak{sp}_4 \simeq \mathfrak{so}_5$

$m=3$: $\mathfrak{g}(A) \simeq \mathfrak{g}_2$

$m=4$: $\mathfrak{g}(A)$ - "twisted version" of $\mathfrak{sl}_2$ $= A_2^{(2)}$ $\leftarrow$ we will not discuss those in our course

$m\ge 5$: $\mathfrak{g}(A)$ - too big (has exponential growth) $\Leftarrow$ we shall not discuss those in class.

Def 3: If A-gen. Cartan $\Rightarrow$ $\boxed{\mathfrak{g}(A) \text{ - symmetrizable Kac-Moody alg.}}$

$\boxed{\text{Thm 3 (Gabber - Kac):}}$ For a Kac-Moody alg. $\mathfrak{g}(A)$, the kernel

$$\underline{\mathrm{Ker}\left(\tilde{\mathfrak{g}}(A) \longrightarrow \mathfrak{g}(A)\right)}$$

as an ideal is generated by

$$\boxed{\Big\{ \underbrace{\mathrm{ad}(e_i)^{1-a_{ij}}(e_j)}_{\underbrace{[e_i,[\ldots[e_i,[e_i,e_j]]]]}_{1-a_{ij}}}, \quad \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) \;\Big|\; i \neq j \Big\}}$$

$\boxed{\text{Alternatively}}$: $\mathfrak{g}(A)$ is gen-d by $\{e_i, f_i, h_i\}_{i=1}^{n}$ subject to defining rel-s (1) together with Serre rel-s

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$\boxed{\text{So: For generalized Cartan matrices } A, \ \mathfrak{g}(A) \text{ can be explicitly defined by generators \underline{and} relations, similar to the case of simple f.d. Lie alg-s}}$

⑦

Partial Proof of Thm 3

We will only show that given el-s are indeed in the kernel.

▶ Let's show that $ad(f_i)^{1-a_{ij}}(f_j) \in \ker(\cdot -)$

$$\Updownarrow$$

$$\boxed{ad(f_i)^{1-a_{ij}}(f_j) = 0 \text{ in } \mathfrak{g}(A)} \quad (\dagger)$$

$$\boxed{\textbf{Claim:}\ [e_k, ad(f_i)^{1-a_{ij}}(f_j)] = 0 \quad \forall k.}$$

$\Downarrow$ the ideal gen-d by $ad(f_i)^{1-a_{ij}}(f_j)$ is not intersecting $\mathfrak{h}$

(b/c it lives in $Q_-\setminus\{0\}$ - degrees) $\xrightarrow{(3)}$ must be a zero ideal

$$\Downarrow$$

$$\boxed{ad(f_i)^{1-a_{ij}}(f_j) = 0}$$

So: Claim implies $(\dagger)$.

Proof of Claim

Case 1: $k \neq i, j$

$[e_k, f_i] = 0 = [e_k, f_j] \Rightarrow [e_k, ad(f_i)^{1-a_{ij}}(f_j)] = 0$ ✓

Case 2: $k = j$

$[e_j, f_i] = 0, \ [e_j, f_j] = h_j \Rightarrow [e_j, ad(f_i)^{1-a_{ij}}(f_j)] = \underbrace{ad(f_i)^{1-a_{ij}}(h_j)}_{\text{we claim to be } 0.}$

• If $a_{ij}<0 \Rightarrow a_{ij}\le -1 \Rightarrow [f_i,\underbrace{[f_i,h_j]}_{a_{ji}\cdot f_i}]=0 \Rightarrow \mathrm{ad}(f_i)^{1-a_{ij}}(f_j)=0$

• If $a_{ij}=0 \overset{\nearrow a_{ji}=0}{} \Rightarrow [f_i,h_j]=a_{ji}\cdot f_i=0$

<u>Case 3</u> : $k=i$ (in this case the direct argument is not feasable
Instead : use sl_2- theory)

$$sl_2^{(i)} \subset \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{g} \ni f_j$$

Consider $\underline{sl_2^{(i)}=\langle e_i,h_i,f_i\rangle}$

$$\left.\begin{aligned} i\neq j \Rightarrow [e_i,f_j]&=0\\ [h_i,f_j]&=-a_{ij}f_j \end{aligned}\right\} \Rightarrow f_j\text{ - h.wt vector for } sl_2^{(i)} \Rightarrow \mathrm{ad}(f_i)^{1-a_{ij}}f_j \text{ - singular vector.}$$

$$\Updownarrow$$

$$\boxed{[e_i,\ \mathrm{ad}(f_i)^{1-a_{ij}}f_j]=0}$$

<u>Note</u>: don't claim here $\underline{\mathrm{ad}(f_i)^{1-a_{ij}}f_j \text{ is ZERO}}$ yet!
nevertheless it
follows at the end.

Def 4: A generalized Cartan matrix A is ~~affine~~ if $D \cdot A \geq 0$, but $DA \not> 0$

positive semidefinite. not positive definite.

$\boxed{\text{Rmk}}$: $\det(A) = 0$ for A as in Def 4.

Def 5: A - affine $\Rightarrow$ $\boxed{\mathfrak{g}(A) - \text{affine Kac-Moody}}$

! This agrees with our construction of Lecture 1 !

untwisted affine alg-s.

Thm 4: $\mathfrak{g}$ - simple f.d. Lie alg, i.e. $\mathfrak{g} \simeq \mathfrak{g}(A)$, A - Cartan $n \times n$ matrix.

$\rightsquigarrow \hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$ from Lecture 1 is indeed an affine Kac-Moody alg.

with corresponding matrix $\hat{A}$ being of the form

$$\hat{A} = \left(\begin{array}{c|ccc} 2 & * & \cdots & * \\ \hline * & & & \\ \vdots & & A & \\ * & & & \end{array}\right)$$

$$\begin{array}{ccc} A & \rightsquigarrow & \hat{A} \\ \updownarrow & & \updownarrow \\ \mathfrak{g}\text{-simple f.d} & \rightsquigarrow & \hat{\mathfrak{g}} \end{array}$$

We shall prove this Thm next time.

Rmk: It's exactly the fact that $\hat{\mathfrak{g}}$ may be realized both combinatorially via $\hat{A}$ as well as $L\mathfrak{g} + \mathbb{C}K$ that makes their theory so rich.

Dynkin diagrams for (untwisted affine Kac-Moody) $\simeq \hat{\mathfrak{g}}$

1) $\mathfrak{g} = \mathfrak{sl}_n = \mathfrak{g}(A_{n-1})$, $n \geq 2$

$$A_{n-1} = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix}_{n-1 \times n-1}$$

n-1 vertices

$\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}_n} = \mathfrak{g}(A_{n-1}^{(1)})$

$n=2 \rightsquigarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$$n>2 \rightsquigarrow \left(\begin{array}{c|cccc} 2 & -1 & 0 & \cdots & -1 \\ \hline -1 & & & & \\ 0 & & & & \\ \vdots & & A_{n-1} & & \\ 0 & & & & \\ -1 & & & & \end{array}\right)$$

2) $\mathfrak{g} = \mathfrak{so}_{2n+1} = \mathfrak{g}(B_n)$, $n \geq 3$

$$B_n : \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ 0 & & & -2 & 2 \end{pmatrix}$$

n vertices

$\hat{\mathfrak{g}} = \widehat{\mathfrak{so}_{2n+1}} = \mathfrak{g}(B_n^{(1)})$

$$B_n^{(1)} : \left(\begin{array}{c|cccc} 2 & 0 & -1 & 0 & \cdots \\ \hline 0 & & & & \\ -1 & & & & \\ 0 & & B_n & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)$$

3) $\mathfrak{g} = \mathfrak{sp}_{2n} = \mathfrak{g}(C_n)$, $n \geq 2$

$$C_n : \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -2 \\ 0 & & & -1 & 2 \end{pmatrix}$$

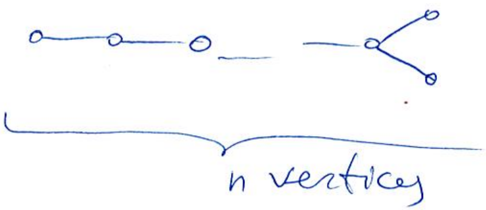
n vertices

$\hat{\mathfrak{g}} = \widehat{\mathfrak{sp}_{2n}} = \mathfrak{g}(C_n^{(1)})$

$$C_n^{(1)} : \left(\begin{array}{c|ccc} 2 & -1 & & \\ \hline -2 & & & \\ 0 & & C_n & \\ & & & \end{array}\right)$$

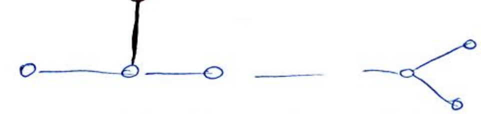
4) $\mathfrak{g} = SO_{2n} = \mathfrak{g}(D_n)$, $n \geq 4$

D_n: $\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & 0 & \\ & \ddots & \ddots & & & \\ & 0 & & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$



$\hat{\mathfrak{g}} = \widehat{SO}_{2n} = \mathfrak{g}(D_n^{(1)})$

$D_n^{(1)}$: $\left(\begin{array}{c|c} 2 & 0 \quad -1 \quad 0 \\ \hline \begin{matrix} 0 \\ -1 \\ 0 \end{matrix} & D_n \end{array}\right)$



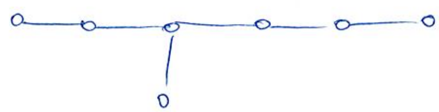
Those were all classical series (ABCD)

But there are also 5 more exceptional types of simple f.d. $\mathfrak{g}$

5) $\mathfrak{g} = \mathfrak{g}(E_6)$

$\hat{\mathfrak{g}} = \mathfrak{g}(E_6^{(1)})$

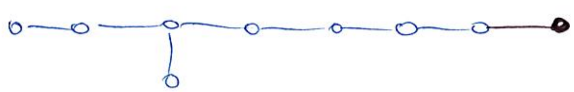
6) $\mathfrak{g} = \mathfrak{g}(E_7)$



$\hat{\mathfrak{g}} = \mathfrak{g}(E_7^{(1)})$

7) $\mathfrak{g} = \mathfrak{g}(E_8)$

$\hat{\mathfrak{g}} = \mathfrak{g}(E_8^{(1)})$



8) $\mathfrak{g} = \mathfrak{g}(F_4)$

$\hat{\mathfrak{g}} = \mathfrak{g}(F_4^{(1)})$

9) $\mathfrak{g} = \mathfrak{g}(G_2)$

$\hat{\mathfrak{g}} = \mathfrak{g}(G_2^{(1)})$

Exceptional types

Hwk Problem: Verify all this data.

Note: There also exist twisted affine Kac-Moody alg-s, but we will not discuss these.