04/06/2021

Lecture #20

Last time : • Contragredient Lie alg-s

generated by $\{e_i, f_i, h_i\}_{i=1}^n$

$$\underline{A} \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \rightsquigarrow \boxed{\tilde{\mathfrak{g}}(A)} \leftarrow Q\text{-graded}$$

$$\tilde{\mathfrak{g}}(A) = \bigoplus_{\alpha\in Q} \tilde{\mathfrak{g}}(A)_\alpha. \qquad Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$$

• A - generalized Cartan matrix $\rightsquigarrow$ $\mathfrak{g}(A)$ - (symmetrizable) Kac-Moody alg.

We Proved: $\{\mathrm{ad}(e_i)^{1-a_{ij}}(e_j),\ \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) \mid i\neq j\} \subseteq \mathrm{Ker}\left(\underline{\tilde{\mathfrak{g}}}(A) \to \mathfrak{g}(A)\right)$

Lie alg. gen-d by $\{e_i, f_i, h_i\}_{i=1}^n$ subject only to rel-s from property (1).

Thm (Gabber - Kac) : They generate the kernel.

$\Downarrow$

! Cor: You can define Kac-Moody $\mathfrak{g}(A)$ by generators & relations.

- Generalized Cartan matrix A is "affine" if $DA \geq 0$, but $DA \not> 0$.

$$\Downarrow$$

$$\boxed{\mathfrak{g}(A) - \text{affine Kac-Moody algebra.}}$$

Last time we stated but didn't prove:

$\boxed{\text{Thm}}$ • $\mathfrak{g}$ - simple f.d. Lie alg w/ Cartan matrix A $\rightsquigarrow$ $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$. (from Lecture 1).

$\uparrow$ is affine Kac-Moody in the above sense

• The gen. Cartan matrix of $\hat{\mathfrak{g}}$ is $\begin{pmatrix} 2 & \cdots \\ \vdots & A \end{pmatrix}$

▶ Pick a Cartan subalg. $\mathfrak{h} \subseteq \mathfrak{g}$, let $r = \mathrm{rk}(\mathfrak{g}) = \dim(\mathfrak{h})$.

Let $\{e_i, h_i, f_i\}_{i=1}^{r}$ be the Chevalley generators for $\mathfrak{g}$ (here we view $\mathfrak{g} = \mathfrak{g}(A)$)

Recall: $\theta \in \Delta_+$ - the maximal/highest root in Δ_+

Consider:

$$\boxed{e_0 := f_\theta \cdot t, \quad f_0 := e_\theta \cdot t^{-1}, \quad h_0 := K - h_\theta, \quad h_\theta = [e_\theta, f_\theta]}$$

Example: $\mathfrak{g} = \mathfrak{sl}_N \Rightarrow e_\theta = E_{1N} = \begin{pmatrix} & 1 \\ & \end{pmatrix}$, $f_\theta = \begin{pmatrix} & \\ 1 & \end{pmatrix} \Rightarrow e_0 = \begin{pmatrix} & \\ t & \end{pmatrix}$, $f_0 = \begin{pmatrix} & t^{-1} \\ & \end{pmatrix}$

Claim: $\{e_i, f_i, h_i\}_{i=0}^{r}$ – these are exactly realizing $\hat{\mathfrak{g}}$ as the contragredient Lie alg. $\mathfrak{g}(\hat{A})$ ← yet to be determined

$*$ First: these el-s generate $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}\cdot K$.

Clearly, they generate $\mathfrak{g} \oplus \mathbb{C}\cdot K$.

Look at $\mathfrak{g}\cdot t\,\mathbb{C}[t]$ (resp. $\mathfrak{g}\cdot t^{-1}\mathbb{C}[t^{-1}]$), so that $L\mathfrak{g} = \mathfrak{g}\cdot t\,\mathbb{C}[t] \oplus \mathfrak{g} \oplus \mathfrak{g}\cdot t^{-1}\mathbb{C}[t^{-1}]$.

They are clearly generated by $\mathfrak{g}\cdot t$ (resp. $\mathfrak{g}\cdot t^{-1}$), b/c. $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

BUT: As $\mathfrak{g}$-modules: $\mathfrak{g}\cdot t \simeq \mathfrak{g}$ ← the adjoint repr-n, $\mathfrak{g}\cdot t^{-1} \simeq$, e_θ, f_θ - generate $\mathfrak{g}$ under the adjoint action

$\Rightarrow$ done! (i.e. $\hat{\mathfrak{g}}$ is gen-d by $\{e_i, f_i, h_i\}_{i=0}^{r}$) ✓

$*$ Next: Check rel-s: $[e_i, f_j] = \delta_{ij} h_i$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$ $\forall i,j \in \{0,\dots,r\}$

$[h_i, h_j] = 0$ ← OBVIOUS

if $i,j \neq 0 \Rightarrow$ they follow automatically (as they are satisfied inside $\mathfrak{g}$).

Let's look now at $[h_i, e_j]$ with one (or both) of $\{i,j\}$ being zero.

- $[h_0, e_j] \overset{?}{=} \boxed{a_{0j}} \cdot e_j \quad \forall\, 1 \le j \le r$ ← yet to be determined!

$\quad \|$

$[K - h_\theta, e_j] = -[h_\theta, e_j] = -\alpha_j(h_\theta)\cdot e_j \Rightarrow a_{0j} = -\alpha_j(h_\theta) = -\left(\alpha_j, \theta^\vee = \frac{2\theta}{\underbrace{(\theta,\theta)}_{=2 \text{ by our choice of } (\cdot,\cdot)}} = \theta\right) = -(\alpha_j, \theta)$

So: $\boxed{a_{0j} = -(\alpha_j, \theta)}_{\forall 1 \le j \le r}$ $\qquad \mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$

Obviously an integer! $\qquad = \theta - (\alpha_j, \theta)\cdot\alpha_j^\vee$

! But it's also non-positive! (b/c otherwise $s_{\alpha_j}(\theta) > \theta$ – contradicting θ-max!)

- $[h_0, e_0] \overset{?}{=} 2e_0$

$\| $

$[K - h_\theta, f_\theta t] = -[h_\theta, f_\theta t] = \theta(h_\theta) \cdot f_\theta t = \underbrace{(\theta, \theta^\vee)}_{=2} \cdot f_\theta t = 2 f_\theta t = 2e_0.$

- $[h_i, e_0] = [h_i, f_\theta t] = -\theta(h_i) \cdot \underbrace{f_\theta t}_{e_0}$ ($i \neq 0$)

$\| ?$

$\boxed{a_{i0}} \cdot e_0$

yet to be determined!

$\Rightarrow a_{i0} = -\theta(h_i) = -(\alpha_i^\vee, \theta) = -\left(\frac{2\alpha_i}{(\alpha_i, \alpha_i)}, \theta\right)$

So: $\boxed{a_{i0} = -(\alpha_i, \theta) \cdot \frac{(\theta, \theta)}{(\alpha_i, \alpha_i)} \quad \forall\, 1 \le i \le r}$

this ratio equals 1, 2, or 3.

$\boxed{\text{Upshot}}$: $a_{0i}, a_{i0} \in \mathbb{Z}_{\le 0}$ $(\forall i \neq 0)$, **AND** $a_{i0} = 0 \Leftrightarrow a_{0i} = 0$.

$*$ $[h_i, f_j] = -a_{ij} \cdot f_j$ — ANALOGOUS (exercise!)

Remains to compute $[e_i, f_j]$ with at least one of $\{i, j\}$ being ZERO.

- $[e_0, f_0] \overset{?}{=} h_0$

$\|$

$[f_\theta \cdot t, e_\theta \cdot t^{-1}] = \underbrace{-h_\theta}_{[f_\theta, e_\theta] = -h_\theta} + K \cdot \underbrace{(f_\theta, e_\theta)}_{=1} = K - h_\theta = h_0.$

- $[e_0, f_i] \overset{?}{=} 0$ ($i \neq 0$)

$\|$

$[f_\theta t, f_i] = \underbrace{[f_\theta, f_i] \cdot t}_{\text{in degree } -\theta - \alpha_i \Rightarrow \text{ it's zero } (\theta \text{ as max root})} = 0.$

- $[e_i, f_0] = 0$ same way! ($i \neq 0$)

$\checkmark$

* Second property of the contragredient alg. is that $\hat{\mathfrak{g}}$ is $\hat{Q}$-graded with

$$\hat{Q} = \bigoplus_{i=0}^{r} \mathbb{Z}\alpha_i = \mathbb{Z}\alpha_0 \oplus Q$$

$$\boxed{\hat{\mathfrak{g}}_0 = \bigoplus_{i=0}^{r} \mathbb{C}h_i, \quad \hat{\mathfrak{g}}_{\alpha_i} = \mathbb{C}e_i, \quad \hat{\mathfrak{g}}_{-\alpha_i} = \mathbb{C}f_i} \qquad (*)$$

Exercise: Check this!

Set $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}\cdot K$. Let $\alpha_0 := \delta - \theta \in \hat{\mathfrak{h}}^*$, where $\underbrace{\delta_{|\hat{\mathfrak{h}}} = 0}$ & $\alpha(K) = 0$ $\forall \alpha \in \mathfrak{h}^*$.

Actually, to make sense of this we already need to work with $\tilde{\mathfrak{g}} = \mathbb{C}d \ltimes \hat{\mathfrak{g}}$, so that $\delta_{|\hat{\mathfrak{h}}} = 0$, $\delta_{|d} = 1$.

Note: $[h, e_0] = \alpha_0(h)\cdot e_0$ & $[h, f_0] = -\alpha_0(h)\cdot f_0$ $\forall h \in \hat{\mathfrak{h}}$

($\text{for } 1 \le i \le r: \ \alpha_0(h_i) = -\theta(h_i) = a_{i0}$; for $i = 0$: $\alpha_0(h_0) = (\delta - \theta)(h_0) = -\theta(h_0) = -\theta(K - h_0) = (\theta, \theta^\vee) = 2$)

Then: $\hat{\mathfrak{g}}$ is $\hat{Q}$-graded with $\deg(e_i) = \alpha_i = -\deg(f_i)$, $\deg(h_i) = 0$

! Explicitly, this $\hat{Q}$-grading is defined via:

$$\boxed{\deg(K) = 0, \quad \deg(x t^n) = \deg_Q(x) + n\delta \quad \forall n \in \mathbb{Z}, x \in \mathfrak{g}}$$

in particular, $(*)$ is clear.

* Third Property of the contragredient Lie algebras is that every $\hat{Q}$-graded ideal intersects nontrivially the degree ZERO part.

Assume the contrary, i.e. $\exists\ \{0\} \neq I \subseteq \hat{\mathfrak{g}}$ - $\hat{Q}$-graded ideal, s.t. $I \cap \hat{\mathfrak{h}} = 0$.

Consider the image $\bar{I} \subseteq L\mathfrak{g}$ of I under the natural projection $\hat{\mathfrak{g}} \xrightarrow{K \mapsto 0} L\mathfrak{g}$

$\left.\begin{array}{l} \underline{\text{Then}}: \bar{I} - \hat{Q}\text{-graded, nonzero ideal of } L\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g} \cdot t^n \\ \underline{\text{BUT}}: \hat{Q}\text{-degrees appearing in } \mathfrak{g} \cdot t^n \ \&\ \mathfrak{g} \cdot t^m \ (n \neq m) \text{ are pairwise distinct} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} \Rightarrow \exists\ a \in \mathfrak{g} \setminus \{0\},\ \exists n \in \mathbb{Z} \text{ s.t. } a \cdot t^n \in \bar{I} \\ \underline{\text{However}}: \mathfrak{g}\text{-simple} \Rightarrow \mathrm{ad}(\mathfrak{g})(a) = \mathfrak{g} \end{array}\right\} \Rightarrow \bar{I} = L\mathfrak{g} \Rightarrow \mathfrak{h} \in \bar{I} \Rightarrow I \cap \hat{\mathfrak{h}} \neq 0 \Rightarrow \lightning$

Contradiction!

This completes our proof of the Theorem! ∎

Remark: (a) $\hat{A}$ - indecomposable (can be verified case-by-case)

(b) A - Cartan $\Rightarrow$ symmetric, i.e. $(D \cdot A)^T = D \cdot A$ with $D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \ d_r \end{pmatrix}$, $d_i = \frac{(\alpha_i, \alpha_i)}{2}$, $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$

Claim: $\hat{A}$ - symmetrizable.

▸ Consider $\hat{D} = \begin{pmatrix} 1 & & & 0 \\ & d_1 & & \\ & & d_2 & \\ 0 & & & \ddots \ d_r \end{pmatrix} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & D \end{array}\right)$. Claim that $(\hat{D} \cdot \hat{A})^T = \hat{D} \cdot \hat{A}$.

For $i,j \neq 0$, $(\hat{D}\hat{A})_{ij} = (\hat{D}\hat{A})_{ji}$ follows from $(DA)_{ij} = (DA)_{ji}$.

Finally, for $i=0, j\neq 0$: $(\hat{D}\hat{A})_{0j} = a_{0j} = -(\alpha_j, \theta)$

$$(\hat{D}\hat{A})_{j0} = d_j a_{j0} = \frac{\cancel{(\alpha_j,\alpha_j)}}{\cancel{2}} \cdot \frac{-(\alpha_j,\theta)\cdot \cancel{2}}{\cancel{(\alpha_j,\alpha_j)}} = -(\alpha_j,\theta)$$

▪

(c) Let's express θ as a linear combination $\theta = \sum_{i=1}^{r} a_i \alpha_i$, $a_i \in \mathbb{Z}_{\geq 0}$ (actually positive!)

Set $\boxed{\delta := \alpha_0 + \sum_{i=1}^{r} a_i \alpha_i}$

It follows from (b) and proof of Thm that $(\delta, \alpha_i) = 0$ $\forall i$

Equivalently, the linear combination of the columns of $\hat{D}\hat{A}$ with coeff-s $\{1, a_1, a_2, \ldots, a_r\}$ is ZERO.

Moreover, δ spans the kernel of $\hat{D}\hat{A}$.

Def: The roots of the contragredient $\mathfrak{g} = \mathfrak{g}(A)$ are elements of

$$\Delta = \{ \alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$$

Rmk: (a) The assignment $e_i \mapsto f_i$, $f_i \mapsto e_i$, $h_i \mapsto -h_i$ gives rise to an algebra autom. of $\tilde{\mathfrak{g}}(A)$, hence, also of $\mathfrak{g}(A)$. Therefore:

$$\boxed{\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}}$$

(b) For any $\alpha \in \Delta_+$, $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ is spanned by $[e_{i_1}, [e_{i_2}, \ldots, [e_{i_{r-1}}, e_{i_r}] \ldots]]$ with $\alpha_{i_1} + \ldots + \alpha_{i_r} = \alpha$

$\Downarrow$

$$\boxed{\dim \mathfrak{g}_\alpha < \infty \quad \forall \alpha \in \Delta}$$

Finally, for $\hat{\mathfrak{g}}$ (where $\mathfrak{g}$ - simple f.dim), as follows from the proof of the theorem, we have:

* Root Decomposition:

$$\boxed{\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \bigoplus_{\substack{(\alpha,k)\neq(0,0) \\ \alpha\in\Delta\amalg\{0\} \\ k\in\mathbb{Z}}} \mathfrak{g}_\alpha \cdot t^k \qquad \text{with } \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K, \quad \Delta = \Delta(\mathfrak{g})}$$

* Root system:

$$\boxed{\Delta(\hat{\mathfrak{g}}) = \Delta(\mathfrak{g}) \amalg \coprod_{k\in\mathbb{Z}\setminus\{0\}} \{\alpha + k\delta \mid \alpha\in\Delta(\mathfrak{g})\amalg\{0\}\}}$$

* Positive roots

$$\boxed{\Delta_+(\hat{\mathfrak{g}}) = \Delta_+(\mathfrak{g}) \amalg \coprod_{k>0} \{\alpha + k\delta \mid \alpha\in\Delta(\mathfrak{g})\amalg\{0\}\}}$$

Define

$$\boxed{F := Q \otimes_{\mathbb{Z}} \mathbb{C}}$$ - v. space / $\mathbb{C}$ with basis $\{\alpha_1, \ldots, \alpha_n\}$

Def: Consider the linear operator $F \to \mathfrak{h}^*$, $\alpha \mapsto \bar{\alpha}$, defined so that

$$\boxed{\bar{\alpha}_j(h_i) = a_{ij} \quad \forall\, i,j \in \{1, \ldots, n\}}$$

This construction satisfies

$$\boxed{[h, x] = \bar{\alpha}(h) \cdot x \quad \forall h \in \mathfrak{h} \quad \forall x \in \mathfrak{g}_\alpha \quad (\alpha \in \Delta)}$$

Rmk: (a) Above $F \to \mathfrak{h}^*$ is isom iff A - nondegenerate (e.g. A - Cartan)

(b) For $\mathfrak{g}(A) = \hat{\mathfrak{g}}$, the kernel of $F \to \mathfrak{h}^*$ is 1-dim, spanned by $\underline{\underline{\delta}}$.

Next time, we shall discuss category $\mathcal{O}$ for contragredient $\mathfrak{g}(A)$.

Today: Warm up by recalling this for simple f.dim. $\mathfrak{g} = \mathfrak{g}(A)$

Def: The category $\mathcal{O}$ of modules over $\mathfrak{g} = \mathfrak{g}(A)$ is defined as follows:

Obj $(\mathcal{O})$: $\mathfrak{g}$-modules M satisfying 3 conditions:

(1) M is $\mathfrak{h}$-diagonalizable: $M = \bigoplus_{\mu \in \mathfrak{h}^*} M[\mu]$, $M[\mu] = \{v \in M \mid h(v) = \mu(h) \cdot v \ \ \forall h \in \mathfrak{h}\}$

(2) $\dim(M[\mu]) < \infty \quad \forall \mu$

(3) $\exists$ finite set $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$, s.t.

$$\mathrm{Supp}(M) \subseteq \bigcup_{i=1}^{m} D(\lambda_i)$$

where

$$\mathrm{Supp}(M) := \{\mu \in \mathfrak{h}^* \mid M[\mu] \neq 0\}$$

$$D(\lambda) := \{\lambda - n_1\alpha_1 - \ldots - n_r\alpha_r \mid n_i \in \mathbb{Z}_{\geq 0}\} \subseteq \mathfrak{h}^*$$

Mor $(\mathcal{O})$: $\mathfrak{g}$-module homomorphisms ← NOTE: Clearly they map $M[\mu] \to N[\mu]$

Rmk: This can be viewed as a refinement of our discussion in the 1st month of the course (BUT now everything is graded not just by $\mathbb{Z}$ but rather by a lattice Q).

In particular, $\forall \lambda \in \mathfrak{h}^* \rightsquigarrow$ have Verma M_λ, its irreducible quot-t L_λ

Clearly: $M_\lambda, L_\lambda \in \mathcal{O}$.

Also: Any graded submodule, or a quotient by a graded submodule, of $M \in \mathcal{O}$ is again in $\mathcal{O}$.

Def: For $M \in \mathcal{O}$, define its *formal character*

$$\boxed{ch(M) = \sum_{\mu \in \mathfrak{h}^*} \dim(M[\mu]) \cdot e^\mu}$$

it's an elt of the ring $R = \{ \sum_{\mu \in \mathfrak{h}^*} \overset{\mathbb{Z}_{\geq 0}}{\overset{\in}{a_\mu}} e^\mu \mid$ supported at finite union of $D(\lambda)$'s $\}$

Example 1: $$\boxed{ch(M_\lambda) \overset{PBW}{=} \frac{e^\lambda}{\prod_{\alpha \in \Delta_+} (1-e^{-\alpha})}}$$ counted with multiplicity

Example 2: $A=(2)$, i.e. $\mathfrak{g}(A) \simeq \mathfrak{sl}_2$. Identify $\mathfrak{h}^* \simeq \mathbb{C}$ via $\alpha \mapsto 2$, or $\omega = \frac{\alpha}{2} \mapsto 1$. Set $x := e^{\omega_1}$.

$$\Rightarrow \boxed{ch(M_\lambda) = \frac{x^\lambda}{1-x^{-2}}}$$

If $\lambda \in \mathbb{Z}_{\geq 0}$, then $ch(L_\lambda) \overset{\text{see Lemma below}}{=} ch(M_\lambda) - ch(M_{-\lambda-2}) = \dfrac{x^{\lambda+1} - x^{-\lambda-1}}{x - x^{-1}}$ ← This is the simplest example of Weyl-Kac formula

(due to $0 \to M_{-\lambda-2} \to M_\lambda \to L_\lambda \to 0$.)

which is also clear from the $\mathfrak{sl}_2$-weights of L_λ being $\{\lambda, \lambda-2, \dots, -\lambda+2, -\lambda\}$ with dim $= 1$.

Lemma 1: (a) $M_1, M_2 \in \mathcal{O} \Longrightarrow M_1 \otimes M_2 \in \mathcal{O}$ and

$$\boxed{ch(M_1 \otimes M_2) = ch(M_1) \cdot ch(M_2)}$$

(b) $0 \to N \to M \to M/N \to 0$ - short exact sequence in $\mathcal{O}$

$$\Downarrow$$

$$\boxed{ch(M) = ch(N) + ch(M/N)}$$

▶ (a) Follows from $(M_1 \otimes M_2)[\mu] = \bigoplus_{\mu_1+\mu_2=\mu} M_1[\mu_1] \otimes M_2[\mu_2]$

(b) Follows from $0 \to N[\mu] \to M[\mu] \to (M/N)[\mu] \to 0$. ◼

Exercise: Provide $M_1, M_2 \in \mathcal{O}$, s.t. $ch(M_1) = ch(M_2)$ **BUT** $M_1 \not\cong M_2$.