Lecture 21

Last time: • $\mathcal{O}$ for simple f.d. $\mathfrak{g} = \mathfrak{g}(A)$ with Cartan matrix A.

$\hookrightarrow$ $M = \bigoplus_{\mu \in \mathfrak{h}^*} M[\mu]$, $\quad M[\mu]$ - fin. dim.

Support of $M = \{\mu : M[\mu] \neq 0\} \subseteq \bigcup_{i=1}^{m} D(\lambda_i)$

finite union

with $D(\lambda) = \{\lambda - n_1\bar{\alpha}_1 - \dots - n_r\bar{\alpha}_r \mid n_i \in \mathbb{Z}_{\geq 0}\}$

$\bar{\alpha} \in \mathfrak{h}^*$

Q: Can this be generalized to any Kac-Moody?

Warning: $\dim(M[\mu])^{<\infty}$ is spoiled if don't make any changes.

e.g. look at $\hat{\mathfrak{g}} = \mathfrak{g}(A)$
$$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}\cdot K$$

Look at the Verma module M_λ: the problem is

$$(h \cdot t^k)(\underbrace{v_\lambda}_{\text{h.wt vector of } M_\lambda}) \in M_\lambda[\lambda] \qquad \forall h \in \mathfrak{h} \subseteq \mathfrak{g}, \ \forall k < 0$$

$$\Downarrow$$

$$\boxed{\dim(M_\lambda[\lambda]) = \infty}$$

But we do want our category to contain all Vermas, hence, their irred. quot-s

Def: Let $A \in \mathrm{Mat}_{r\times r}(\mathbb{C}) \rightsquigarrow \mathfrak{g}(A)$ - corresp. contragredient Lie alg.

Define

extended algebra: $\boxed{\mathfrak{g}_{ext}(A) := \mathfrak{g}(A) \oplus \mathbb{C}\cdot D_1 \oplus \dots \oplus \mathbb{C}\cdot D_r} = \boxed{\mathbb{C}^r \ltimes \mathfrak{g}(A)}$

$\mathbb{C}^r$ has a basis $\{D_i\}_{i=1}^r$

with $[D_i, D_j] = 0$, $[D_i, e_i] = e_i$, $[D_i, f_i] = -f_i$, $[D_i, h_i] = 0$ $\forall i$

$[D_i, e_j] = [D_i, f_j] = [D_i, h_j] = 0$ $\forall i \neq j$.

Note: $\mathfrak{g}_{ext}(A) = \mathfrak{n}_- \oplus \mathfrak{h}_{ext} \oplus \mathfrak{n}_+$, $\mathfrak{h}_{ext} = \mathfrak{h} \oplus \mathbb{C}D_1 \oplus \dots \oplus \mathbb{C}\cdot D_r$.

$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

$\boxed{\dim \mathfrak{h}_{ext} = 2 \dim \mathfrak{h}}$

So: $\mathfrak{g}_{ext}(A)$ has a twice larger Cartan subalgebra!

Recall from last time:

- Last time: $F = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{i=1}^{r} \mathbb{C}\alpha_i$

$F \longrightarrow \mathfrak{h}^*$, $\alpha \longmapsto \bar{\alpha}$ $\underbrace{\text{determined by } \bar{\alpha}_j(h_i) = a_{ij}}_{\text{designed to satisfy}}$ $\forall i,j$

not isom. unless A-nondeg.

$$\boxed{[\underset{\mathfrak{h}}{h}, \underset{\mathfrak{g}_\alpha}{x}] = \bar{\alpha}(h)\cdot x.}$$

i.e. each α_j is viewed as a functional $\mathfrak{h} \to \mathbb{C}$

- NOW: We will rather view α_j as functionals $\mathfrak{h}_{ext} \to \mathbb{C}$

given by: $\boxed{h_i \longmapsto a_{ij}, \quad D_i \longmapsto \delta_{ij}}$

Hence, a map $\boxed{\underset{= \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{C}}{F} \longrightarrow \mathfrak{h}^*_{ext}}$

Define $\boxed{\underline{P} := F \oplus \mathfrak{h}^* = \bigoplus_{i=1}^{r} \mathbb{C}\alpha_i \oplus \bigoplus_{i=1}^{r} \mathbb{C}h_i^*}$ ← twice bigger than F!

Have a natural map

$$\boxed{\varphi : P \longrightarrow \mathfrak{h}^*_{ext}}$$

s.t. (1) $h_j^* \longmapsto (h_i \mapsto \delta_{ij},\ D_i \mapsto 0)$.

(2) on the F-part – as on previous page

The matrix of φ in the corresp. bases is:

$$\left(\begin{array}{c|c} I & * \\ \hline 0 & I \end{array}\right)$$ – non deg!

Claim: φ – v. space isom!

Upshot: After extending
- $\mathfrak{g}(A) \rightsquigarrow \mathfrak{g}_{ext}(A)$
- $\mathfrak{h}^* \rightsquigarrow \mathfrak{h}^*_{ext}$

we can introduce category $\mathcal{O}$ same way as last time.

At least all Vermas M_λ, hence L_λ's, will be in $\mathcal{O}$.

! **Rmk**: In [Feigin-Zelevinsky], they do not extend $\mathfrak{g}(A)$ itself.

BUT : when they define category $\mathcal{O}$, the have an extra

P-grading : $M = \bigoplus_{\mu \in P} M[\mu]$

(in our case $M = \bigoplus_{\mu \in \mathfrak{h}_{ext}^*} M[\mu]$, $P \simeq \mathfrak{h}_{ext}^*$)

Lemma 1 : $$\boxed{ch(M_\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim(\mathfrak{g}_{-\alpha})}}} \Longleftarrow \text{from PBW}$$

Note : For $\mathfrak{g} = \mathfrak{g}(A)$ - simple f.d, $\dim(\mathfrak{g}_{-\alpha}) = 1 \ \forall \alpha \in \Delta_+ \Rightarrow$ recover classical f-la.

Note: We could apply the same construction for simple f.d. $\mathfrak{g} = \mathfrak{g}(A)$, but we'll basically recover def-n from last time, see next Lemma! from "extended" approach

Lemma 2: $\forall \chi \in \mathfrak{h}^*$, let $\mathcal{O}_\chi \subseteq \mathcal{O}$ - full subcategory of modules M with $\mathrm{supp}(M) \subseteq \chi + F (\subset P)$

(a) $\mathcal{O} = \bigoplus_{\chi \in \mathfrak{h}^*} \mathcal{O}_\chi$ $\quad \longleftarrow \quad e_i, f_i, h_i (M_{[\mu]}) \subseteq M_{[\chi + F]}$

(b) $\forall \chi_1, \chi_2 \in \mathfrak{h}^*$ s.t. $\chi_1 - \chi_2 = \bar{\alpha}$ (Image($F \to \mathfrak{h}^*$)), then cat-s $\mathcal{O}_{\chi_1} \simeq \mathcal{O}_{\chi_2}$.

$\Downarrow$

(c) For simple $\mathfrak{g}$, all $\mathcal{O}_\chi \simeq \mathcal{O}_0$ $\longleftarrow$ $\boxed{\text{defined last time}}$

$M \in \mathcal{O}_{\chi_2} \rightsquigarrow$ define $M' \in \mathcal{O}_{\chi_1}$ via $M'_\alpha := M_{\alpha - \chi_1 + \chi_2 + \alpha}$

$\boxed{\text{Rem}}$: (a) For simple f.d. $\mathfrak{g}$, recover our old $\Theta = \Theta_0$.

(b) For affine Kac-Moody $\mathfrak{g}(A)$, the kernel $(F \to \mathfrak{h}^*)$ is 1-dim spanned by δ $\Rightarrow$ it suffices to add only 1 derivation.

Recall: For $\hat{\mathfrak{sl}}_n \rightsquigarrow \boxed{\tilde{\mathfrak{sl}}_n = \mathbb{C}d \ltimes \hat{\mathfrak{sl}}_n}$

$[d, xt^n] = n \cdot xt^n, \quad [d, K] = 0.$

$\downarrow$

For $\hat{\mathfrak{g}} \rightsquigarrow \boxed{\tilde{\mathfrak{g}} = \mathbb{C}d \ltimes \hat{\mathfrak{g}}}$ (defined by the same formula)

$\boxed{\text{Lemma 3}}$ (a) The center Z of $\mathfrak{g}(A)$ is always in $\mathfrak{h}$ and equals:

$$\underbrace{\left\{ \sum_{i=1}^{r} \beta_i h_i \mid \beta_i \in \mathbb{C},\ \sum \beta_i a_{ij} = 0 \quad \forall j \right\}}$$

$[\Rightarrow \dim(Z) = \dim(\operatorname{Ker} \underline{A})]$

(b) If A - generalized Cartan $\Rightarrow$ $\boxed{[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}}$.

(c) If A is indecomposable symmetrizable matrix, then any proper graded ideal is contained in Z.

$[\Rightarrow \mathfrak{g}(A)$ has no proper ideals if A - nondeg$]$.

(Proof of Lemma)

(a) Pick any central $x \in Z$ $\Big\} \Rightarrow$ every Q-degree component of x is central.

$\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$

$\Rightarrow$ WLOG can assume $x \in \mathfrak{g}_\alpha$.

- If $\alpha \neq 0 \Rightarrow \underbrace{\mathbb{C}x}_{\text{Lie subalg., ideal, it doesn't intersect } \mathfrak{h}} \subseteq \mathfrak{g}$ $\overset{\text{Property (3) of contragredient Lie algebras}}{\Longrightarrow}$ ↯

$\Downarrow$

- α must be $0 \Rightarrow x \in \mathfrak{h} = \bigoplus_{i=1}^{r} \mathbb{C}h_i \Rightarrow x = \underline{\sum \beta_i h_i}$

But: $\begin{cases} [x, e_j] = 0 & \forall j \\ [x, f_j] = 0 & \forall j \end{cases} \iff \sum \beta_i a_{ij} = 0 \quad \forall j.$

This completes part (a) ✓

(b) Want : $\boxed{[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}}$ ← $\mathfrak{g}$ is gen-d by $\{e_i, h_i, f_i\}_{i=1}^{r}$ ⇒ $\boxed{\text{Hence, it suffices to prove } e_i, f_i, h_i \in [\mathfrak{g},\mathfrak{g}]}$

$a_{ii} \neq 0 \Rightarrow h_i = [e_i, f_i]$

$e_i = [h_i, e_i]/a_{ii}$

$f_i = \ldots\ldots$

✓

(c) Pick such an ideal $\boxed{0 \neq I \subsetneq \mathfrak{g}(A)}$ – Q-graded ideal.

$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

$\Rightarrow \underline{I = \underset{\overset{\shortparallel}{I\cap\mathfrak{n}_-}}{I_-} \oplus \underset{\overset{\shortparallel}{I\cap\mathfrak{h}}}{I_0} + \underset{\overset{\shortparallel}{I\cap\mathfrak{n}_+}}{I_+}}$

Assume that $I \not\subseteq Z$!

Then, I_+ or I_- must be nonzero! Assume $I_+ \neq \emptyset$, i.e. $\exists\, \overset{0}{\overset{\shortparallel}{a}} \in (I_+)_\alpha$.

Consider the ideal $J \subseteq I$ generated by the element a.

Def-n of $\mathfrak{g}(A) \Rightarrow \underbrace{J \cap \mathfrak{h}}\neq 0$.

$0 \neq x = ad(f_{i_1})\underbrace{ad(f_{i_2})\ldots\ ad(e_{j_1})\ldots\ ad(e_{j_m})(a)}$

$\Rightarrow x \sim h_{i_1} \Rightarrow \underline{J \ni h_{i_1}}$

Start commuting h_{i_1} with e_j, f_j. $\Big\} \longrightarrow \forall\, e_j, f_j$, hence $h_j = [e_j, f_j]$ are in J.

$\Rightarrow J$

$\Uparrow$

$I = \mathfrak{g}(A)$.

$\Uparrow$

↯

Invariant Form

$A \in \mathrm{Mat}_{r\times r}(\mathbb{C}) \rightsquigarrow \mathfrak{g}(A)$

$$\begin{array}{c}\mathfrak{g}(A)\\ \uparrow\\ \tilde{\mathfrak{g}}(A)\end{array}$$

Goal: Construct symmetric bilinear forms

$$\boxed{\begin{aligned}(\cdot,\cdot) &: \mathfrak{g}(A)\times\mathfrak{g}(A) \longrightarrow \mathbb{C}\\ (\cdot,\cdot) &: \tilde{\mathfrak{g}}(A)\times\tilde{\mathfrak{g}}(A) \longrightarrow \mathbb{C}\end{aligned}}$$

s.t.

* they are invariant.
* of "degree 0", i.e.

$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha+\beta\neq 0$.

NOTE: Do not require it to be nondeg.

Assumption: $\underbrace{(e_i, f_i)}_{d_i} \neq 0 \quad \forall\, 1\le i\le r$ ⟵ "Not VERY degenerate"

• First, for such pairing to exist A must be symmetrizable!

Indeed:

$$(h_i, h_j) = (h_i, [e_j, f_j]) = (\underbrace{[h_i, e_j]}_{a_{ij} e_j}, f_j) = a_{ij} \cdot d_j$$

$$\| $$

$$(h_j, h_i) = \text{-- -- ----} = a_{ji} \cdot d_i$$

$\Rightarrow$ $\boxed{A \cdot \underset{\operatorname{diag}(d_i)}{\underset{\|}{D}} \text{ - symmetric!}}$

$\Rightarrow$ $\boxed{\text{Assume: } A\text{-symmetrizable}}$

$\boxed{\text{Claim}}$: If A is indecomposable symmetrizable, then $\{d_i\}$ are unique up to a common scalar!

Easy Exercise

$\boxed{\text{Thm 1}}$ If A is an indecomposable symmetrizable, then there exists a nonzero symm. inv. form of degree 0 on $\mathfrak{g}(A)$, $\tilde{\mathfrak{g}}(A)$.
(and it is unique up to a nonzero factor!)

Main Thm for today

Rmk: $\tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A)$ $\Longrightarrow$ suffices to treat $\mathfrak{g}(A)$!

$(\cdot,\cdot) \leftsquigarrow (\cdot,\cdot)$

Proof (follows [Feigin – Zelevinsky, pp 51-52])

Construction of $\mathfrak{g}(A) \times \mathfrak{g}(A) \xrightarrow{(\cdot,\cdot)} \mathbb{C}$ is by induction on $k \in \mathbb{N}$

$$\boxed{\mathfrak{g}^k := \bigoplus_{\substack{\alpha \in \Delta \sqcup \{0\} \\ |\alpha| \leq k}} \mathfrak{g}_\alpha}$$

height $\leq k$ elements

$\alpha \in Q$
$\parallel$
$\sum k_i \alpha_i \implies \boxed{|\alpha| = \sum |k_i|.}$ ← "height" of α

We'll show how to construct $(\cdot,\cdot) : \mathfrak{g}^k \times \mathfrak{g}^k \to \mathbb{C}$.

invariant in the sense (*) $([x,y],z) = (x,[y,z])$ as long as $[x,y], z, x, y, [y,z] \in \mathfrak{g}^k$.

Taking $k \to \infty$ will produce the form on $\mathfrak{g}(A)$.

- $\boxed{\text{Step of induction: } k=1}$

$\mathfrak{g}^1$ – spanned by $\{e_i, f_i, h_i\}$

$$\begin{cases} (e_i, f_i) = (f_i, e_i) = d_i \\ (h_i, h_j) = a_{ij} d_j = a_{ji} d_i \\ \text{all other pairings are ZERO.} \end{cases}$$

recovered uniquely (up to common factor) from A·D – symmetric!

← Exercise: check (*).

- **Step of Induction**

Assume we have $\mathfrak{g}^k \times \mathfrak{g}^k \to \mathbb{C}$ and want to extend to $\mathfrak{g}^{k+1} \times \mathfrak{g}^{k+1} \to \mathbb{C}$.

Let $\alpha \in \Delta_+$, s.t. $|\alpha| = k+1$, let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$.

$\boxed{\text{Want}}$: $(x, y) = ?$

Claim: x can be written as $\sum_\ell [a_\ell\, b_\ell]$, $a_\ell, b_\ell \in \mathfrak{g}^k$.

Motivated by the expected "invariance property".

Set:
$$\boxed{(x, y) = (y, x) := \sum_\ell (\underset{\mathfrak{g}^1}{\underset{\cap}{a_\ell}}, \underbrace{[b_\ell, y]}_{\underset{\mathfrak{g}^{-1}}{\cap}})}$$

Need to verify:

(1) it's well defined

(2) invariant in the sense of $(*)$.

$\boxed{(1)}$ $\sum_\ell [a_\ell, b_\ell] = 0 \implies \boxed{\sum (a_\ell, [b_\ell, y]) \overset{?}{=} 0 \qquad \forall y}$

It suffices to treat $y = [u, v]$, $u, v \in \mathfrak{g}^k$

$$(a_\ell, [b_\ell, y]) = (a_\ell, [b_\ell, [u, v]]) \overset{\text{Jacobi}}{=} (a_\ell, [[b_\ell, u], v]) + (a_\ell, [u, [b_\ell, v]])$$

$$\overset{\text{induction}}{=} ([a_\ell, [b_\ell, u]], v) + ([[b_\ell, v], a_\ell], u)$$

$$\overset{\text{induction}}{=} (v, [a_\ell, [b_\ell, u]]) + (v, [[a_\ell, u], b_\ell])$$

$$\overset{\text{Jacobi}}{=} (v, [[a_\ell, b_\ell], u])$$

$$\Rightarrow \sum_\ell (a_\ell, [b_\ell, y]) = (v, [\underbrace{\sum [a_\ell, b_\ell]}_{=0}, u]) = 0 \qquad \checkmark$$

Warning: The key part to check is that whenever we used the induction assumption all el-s were indeed in $\mathfrak{g}^k$ (and NOT in $\mathfrak{g}^{k+1}$)

(2) Remarks to verify $(\cdot,\cdot)$ is invariant

Let $|\alpha| = k+1$.

$\beta \in \Delta^+$, $|\beta| \le k$.

(1) $([x,y],z) \overset{?}{=} (x,[y,z]) \quad \forall\, x \in \mathfrak{g}_{\alpha-\beta},\ y \in \mathfrak{g}_\beta,\ z \in \mathfrak{g}_{-\alpha}$.

► $\alpha - \beta \in \Delta^+$ (otherwise both side are tautologically ZERO!)

Equality follows from construction. ∎

(2) $([x,y],z) \overset{?}{=} (x,[y,z])$ for $x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\beta},\ z \in \mathfrak{g}_{\beta-\alpha}$, $\beta \in \Delta_+$, $|\beta| \le k+1$.

► WLOG assume $x = [a,b]$, $a, b \in \mathfrak{g}^k$.

$$\Rightarrow ([x,y],z) = ([[a,b],y],z) \overset{\text{Jacobi}}{=} ([a,[b,y]],z) + ([[a,y],b],z)$$

$$\overset{\text{Induction}}{=} (a,[[b,y],z]) + ([b,z],[a,y]) = (a,[[b,y],z]) + (a,[y,[b,z]])$$

$$\overset{\text{Jacobi}}{=} (a,[b,[y,z]]) \overset{\text{Ind}}{=} (\underbrace{[a,b]}_{x},[y,z])$$ ✓

Warning: Again need to check all el-s are in $\mathfrak{g}^k$ whenever ind. assumption is used! ∎