04/15/2021

Lecture 22

$\boxed{\text{Last time}}$ $\underbrace{\mathfrak{g}_{ext}(A)}_{\text{extended contragredient algebra}} = \mathfrak{g}(A) \rtimes \underbrace{\mathbb{C}^r}_{r \text{ derivations}}$

root lattice

$Q \otimes_{\mathbb{Z}} \mathbb{C}$

$\boxed{\mathfrak{h}^* \oplus F \xrightarrow{\sim} \mathfrak{h}^*_{ext}}$

Application 1: ϑ

Application 2: Non-degenerate symmetric inv. pairing.

$\boxed{\text{Problem 1 on Hwk 11}}$

Def: Define $\underline{\rho} \in \mathfrak{h}^*$ via $\boxed{\rho(h_i) = \frac{a_{ii}}{2} \ \forall i}$ ← this works for any contragredient Lie alg-s

If $\mathfrak{g}(A)$ – Kac-Moody $\Rightarrow a_{ii} = 2 \ \forall i \Rightarrow \boxed{\rho(h_i) = 1 \ \forall i}$ for Kac-Moody

Explicitly: $\varphi, \psi \in \mathfrak{h}^*$, $\alpha, \beta \in Q$

$$\boxed{(\varphi+\alpha, \psi+\beta) = \varphi(h_\beta) + \psi(h_\alpha) + (h_\alpha, h_\beta)}$$

Recall

$$\boxed{P = \underset{\underset{\rho}{\cup\!\!\!\!|}}{\mathfrak{h}^*} \oplus \underbrace{(Q \otimes_{\mathbb{Z}} \mathbb{C})}_{=:F}}$$

, [Hwk 11; Problem 1]: pairing $(\cdot, \cdot): P \times P \to \mathbb{C}$.

Rmk: $\boxed{(\rho, \rho) = 0}$ (b/c any el-s from $\mathfrak{h}^* \subset P$ pair trivially)

(Two properties of ρ) $\boxed{(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)}$

$$(\rho, \alpha_i) = \rho(h_{\alpha_i}) = \rho\left(\frac{h_i}{d_i}\right) = \frac{a_{ii}}{2d_i}$$

$$(\alpha_i, \alpha_i) = (h_{\alpha_i}, h_{\alpha_i}) = \left(\frac{h_i}{d_i}, \frac{h_i}{d_i}\right) = \frac{1}{d_i^2} \cdot \underbrace{(h_i, h_i)}_{d_i a_{ii}} = \frac{a_{ii}}{d_i}$$

Usual Casimir ($\mathfrak{g} = \mathfrak{g}(A)$ – simple f.d):

$$C = \sum_{a \in B} a^2 = \sum x_i^2 + \sum_{\alpha>0} \underbrace{(f_\alpha e_\alpha + e_\alpha f_\alpha)} = \boxed{\sum x_i^2 + \underbrace{h_{2\rho}}_{=2h_\rho} + 2\sum_{\alpha>0} f_\alpha e_\alpha}$$

B — orthonormal basis w.r.t. invariant pairing

x_i – orthonormal basis of Cartan

Note: $(e_\alpha, f_\alpha) = 1$ w.r.t. our pairing

Warning: For general contragredient, this sum is infinite but we'll see that it gives rise to a well-defined operator on $M \in \mathcal{O}$!

For each $\alpha \in \Delta^+$, choose dual bases

$$\{e_\alpha^{(i)}\} \subseteq \mathfrak{g}_\alpha \qquad \text{s.t.} \quad (e_\alpha^{(i)}, f_\alpha^{(j)}) = \delta_{ij}$$
$$\{f_\alpha^{(i)}\} \subseteq \mathfrak{g}_{-\alpha}$$

Now, we consider any $\mathfrak{g}(A)$!

Know from last time the pairing is of degree ZERO and non-degenerate.

def

$$\Delta_+ := 2 \sum_{\alpha>0} \sum_{i=1}^{\overbrace{\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}}} f_\alpha^{(i)} e_\alpha^{(i)}$$

– well-defined operator $M \circlearrowleft$ $(\forall M \in \mathcal{O})$

$$\Delta_0 := \sum_{x_i\text{ - orthonormal basis}} x_i^2 + \underbrace{h_{2\rho}}_{=2h_\rho} \; : \; M \circlearrowleft$$

def

Def: For any $M \in \mathcal{O}$, define the $\boxed{\text{Casimir operator}}$

$\Delta : M \longrightarrow M$ via

$$\boxed{\Delta = \Delta_+ + \Delta_0}$$

So: Basically, take the usual f-la, normally reorder, and view it as a linear operator!

Theorem 1: (a) The operator Δ commutes with $\mathfrak{g}(A)$-action

(b) On M_λ, Δ acts by $(\lambda, \lambda+2\rho)\cdot \mathrm{Id}_{M_\lambda}$.

[! Note : (b) $\Rightarrow$ same result for any h. wt. module with h. wt $= \lambda$

(a) is the counterpart of $C \in \mathcal{U}(\mathfrak{g})$ being central for simple f.d. $\mathfrak{g}$]

(b)

$$\left.\begin{array}{l}\Delta_+(v_\lambda) = 0 \\ \Delta_0(v_\lambda) = \left(\sum x_i^2 + 2h_\rho\right)(v_\lambda) = (\lambda, \lambda+2\rho)\cdot v_\lambda\end{array}\right\} \Rightarrow \underbrace{\Delta(v_\lambda) = (\lambda, \lambda+2\rho)\, v_\lambda}_{\text{same as for fin. dim. simple}}$$

$\Downarrow$ (a)

$$\boxed{\Delta = (\lambda, \lambda+2\rho)\cdot \mathrm{Id}_{M_\lambda}}$$

(a) $\mathfrak{g}(A)$ is gen-d by $\{e_k, f_k\}$ $\Rightarrow$ suffices to verify that Δ commutes with $\begin{cases} e_k\text{-action} \\ f_k\text{-action} \end{cases}$ $\forall k$

We'll verify e_k's for f_k's – same argument!

some module in $\mathcal{O}$

Pick any $v \in M[\mu]$ $\Rightarrow$ Want: $\boxed{[\underbrace{\Delta}_{\Delta_0+\Delta_+}, e_k](v) \overset{?}{=} 0}$

$$[\Delta_0, e_k](v) = (\mu+\alpha_k, \mu+\alpha_k+2\rho)\cdot e_k v - (\mu, \mu+2\rho) e_k v$$

$$= \underbrace{\big[(2\mu+2\rho, \alpha_k) + (\alpha_k, \alpha_k)\big]}_{\substack{\| \\ (2\mu, \alpha_k) + 2(\alpha_k,\alpha_k) = 2(\mu+\alpha_k, \alpha_k)}} \cdot \overbrace{e_k v}^{\substack{M[\mu+\alpha_k] \\ \Psi}} = \boxed{2 h_{\alpha_k}(e_k v)}$$

Recall: $\rho(\alpha_k) = \frac{1}{2}(\alpha_k, \alpha_k)$

Let's compute $[\Delta_+, e_k] =$

$$= 2 \sum_{\alpha>0}^{i} [f_\alpha^{(i)} e_\alpha^{(i)}, e_k] v = 2 \sum_{\alpha>0}^{i \le \dim \mathfrak{g}_\alpha} \left(f_\alpha^{(i)} \cdot [e_\alpha^{(i)}, e_k] - [e_k, f_\alpha^{(i)}] \cdot e_\alpha^{(i)} \right) v =$$

Note: For $\alpha = \alpha_k$ – simple root, $\dim \mathfrak{g}_{\alpha_k} = 1$ & $\boxed{[e_k, f_k] = h_{\alpha_k}}$

$$= \underbrace{-2 h_{\alpha_k} e_k v}_{\text{this cancels with } [\Delta_0, e_k](v) \text{ above}} + 2\left(\sum_{\alpha}^{i} f_\alpha^{(i)} \cdot [e_\alpha^{(i)}, e_k] - \sum_{\alpha>0,\, \alpha \ne \alpha_k}^{i} [e_k, f_\alpha^{(i)}] \cdot e_\alpha^{(i)} \right) v$$

Remarks: $\sum_{\alpha}^{i} f_\alpha^{(i)} [e_\alpha^{(i)}, e_k] = \sum_{\alpha \neq \alpha_k}^{i} [e_k, f_\alpha^{(i)}] e_\alpha^{(i)}$.

$\Uparrow$

Claim:
(Fix α!)

$$\boxed{\sum_i f_\alpha^{(i)} \otimes [e_\alpha^{(i)}, e_k] = \sum_j [e_k, f_{\alpha+\alpha_k}^{(j)}] \otimes e_{\alpha+\alpha_k}^{(j)}}$$

$\Updownarrow$ [Completely analogous to Lemmas in Lecture 17 (needed for Sugawara)].

Proof of Claim

$$[e_\alpha^{(i)}, e_k] = \sum_j \left([e_\alpha^{(i)}, e_k], f_{\alpha+\alpha_k}^{(j)}\right) \cdot e_{\alpha+\alpha_k}^{(j)}$$

$\Downarrow$

$$\sum_i f_\alpha^{(i)} \otimes [e_\alpha^{(i)}, e_k] = \sum_{i,j} f_\alpha^{(i)} \otimes e_{\alpha+\alpha_k}^{(j)} \cdot \overbrace{\left([e_\alpha^{(i)}, e_k], f_{\alpha+\alpha_k}^{(j)}\right)}^{(e_\alpha^{(i)},\ [e_k, f_{\alpha+\alpha_k}^{(j)}])}$$

$$= \sum_j \Bigg(\underbrace{\sum_i f_\alpha^{(i)} \cdot (e_\alpha^{(i)}; [e_k, f_{\alpha+\alpha_k}^{(j)}])}_{= [e_k, f_{\alpha+\alpha_k}^{(j)}]} \Bigg) \otimes e_{\alpha+\alpha_k}^{(j)} \quad \blacksquare \quad \checkmark$$

This completes our proof of $[\Delta, e_k] = 0$. $\blacksquare$

Exercise: For $\mathfrak{g}(A) = \hat{\mathfrak{g}}$ ($\mathfrak{g}$ - simple f.d), show that

(Next Hwk)

$$\boxed{\Delta = 2(k + h^\vee)(L_0 + d)}$$

level of your repr-n. 0^{th} Sugawara operator

Locally finite & integrable modules

Def: (a) Given a Lie alg. $\mathfrak{g}$, its module V, a vector $v \in V$ is "<u>of finite type</u>"

if $\boxed{\dim(\mathcal{U}(\mathfrak{g})v) < \infty}$

(b) V – <u>loc. finite</u> if each $v \in V$ is of finite type.

[Exercise: V – loc. finite $\iff V = \sum$ fin. dim. $\mathfrak{g}$-modules.]

Def: A module V over Kac–Moody $\mathfrak{g}(A)$ is <u>integrable</u> if it is locally finite w.r.t. <u>each sl_2-triple</u> $\underbrace{sl_2^{(i)}}_{\langle e_i, h_i, f_i \rangle}$

[<u>Rmk</u> (name): "integrable" An sl_2-module M is loc. finite iff

$$M = \bigoplus \underbrace{L_n}_{\text{f.dim. } sl_2\text{-module}} \longleftarrow$$ each can be integrated to the group.]

$\boxed{\text{Prop 1}}$: $\mathfrak{g} = \mathfrak{g}(A)$ is an integrable module over itself (under the adjoint action)

▶ $\mathfrak{g}(A)$ is generated by $\{e_j, f_j\}_{j=1}^{r}$

Claim: Each f_j (likewise, each e_j) is of finite type!

▶ $\underline{\mathfrak{sl}_2^{(i)}}$ $\quad i = j \Rightarrow \mathfrak{sl}_2^{(i)}$-module generated by $f_{j=i}$ is just $\mathfrak{sl}_2^{(i)} \Rightarrow$ f.dim!

$i \neq j \overset{\text{Exercise}}{\Longrightarrow} \mathfrak{sl}_2^{(i)}$-module generated by f_j is $(1 - a_{ij})$-dim

(due to Serre rel-s)

Claim: $x, y \in \mathfrak{g}$ - of fin. type $\Rightarrow [x, y]$ - as well
(Exercise)

The above 2 claims imply the result!

$\boxed{\text{Prop 2}}$: ~~A~~ $\mathfrak{g}(A)$-module V is integrable **iff** there is a set $\{v_j\}_{j\in J}$ of generators of V over $\mathfrak{g}(A)$ s.t. each v_j is of finite type over each $\mathfrak{sl}_2^{(i)}$.

▶ $\Rightarrow$ Obvious: take all $v \in V$ into the set $\{v_j\}_{j\in J}$.

$\Leftarrow$ Let $v \in V$ and pick $i \in \{1, \ldots, r\}$. Want: v — loc. finite / $\mathfrak{sl}_2^{(i)}$.

V is generated by $\{v_j\}$ $\Rightarrow$ choose $v_1, v_2, \ldots, v_N$ s.t.

$$\boxed{v \in \mathcal{U}(\mathfrak{g}) v_1 + \ldots + \mathcal{U}(\mathfrak{g}) v_N =: V' \subseteq V}$$

$\left.\begin{array}{l}\mathcal{U}(\mathfrak{g})\text{-loc.fin.}\\ W\text{-loc.fin.}\end{array}\right\} \Rightarrow \mathcal{U}(\mathfrak{g})\otimes W\text{-also}$
$\Downarrow$
V'-also (as a quotient)

$\Uparrow$

Exercise: $\otimes$ preserves loc. finiteness

Consider

f. dim. $\mathfrak{sl}_2^{(i)}$-module containing all $v_1, v_2, \ldots, v_N$.

$$\boxed{\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) \otimes W & \twoheadrightarrow & V' \ni v \\ \Psi\quad\quad \Psi & & \Psi \\ x \otimes w & \longmapsto & x(w) \end{array}}$$

Remarks: $\mathcal{U}(\mathfrak{g})$ is loc. finite $\mathfrak{sl}_2^{(i)}$-module

$\left(\mathfrak{g}\text{-mod}\right.$ $\overset{\text{PBW}}{\simeq}$ $S(\mathfrak{g}) = \bigoplus_{k\geq 0} S^k(\mathfrak{g})$, $S^k(\mathfrak{g}) \subseteq \underbrace{\mathfrak{g}\otimes\mathfrak{g}\otimes\ldots\otimes\mathfrak{g}}_{k}$

$\mathfrak{g}$ – $\mathfrak{sl}_2^{(i)}$ – loc. finite by Prop 1

loc. finite / $\mathfrak{sl}_2^{(i)}$

Prop 3 Let L_λ be the irreducible h. wt. module / Kac-Moody $\mathfrak{g}(A)$.

$$\boxed{L_\lambda \text{ - integrable} \iff \lambda(h_i) \in \mathbb{Z}_{\geq 0} \quad \forall i}$$

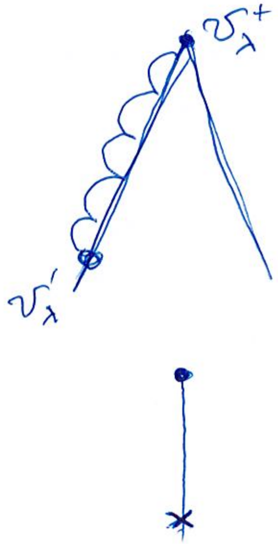
▸ $\boxed{\Rightarrow}$ L_λ - integrable $\Rightarrow$ v_λ^+ - of finite type over $\mathfrak{sl}_2^{(i)} = \langle e_i, f_i, h_i \rangle$ $\forall i$.

Know: $e_i(v_\lambda^+) = 0$

$\Downarrow$ $\mathfrak{sl}_2$-theory

$\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ (otherwise you will not be able to include to f.dim. $\mathfrak{sl}_2^{(i)}$-submodule)

$\boxed{\Leftarrow}$ $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$ $\overset{?}{\Longrightarrow}$ L_λ ? - integrable.



[$\forall i$ Consider: $v_\lambda' = f_i^{1+\lambda(h_i)}(v_\lambda^+)$]

Important construction (shall also recall next time!)

Claim: $e_j(v_\lambda') = 0$ $\forall j$

▸ $j \neq i \Rightarrow [e_j, f_i] = 0 \Rightarrow e_j(v_\lambda') = f_i^{1+\lambda(h_i)} e_j(v_\lambda^+) = 0$.

$j = i$: follows from the $\mathfrak{sl}_2$-theory.

■

(Continuation of the proof)

Thus, we get $v_\lambda' \in L_\lambda$, which is a singular vector,

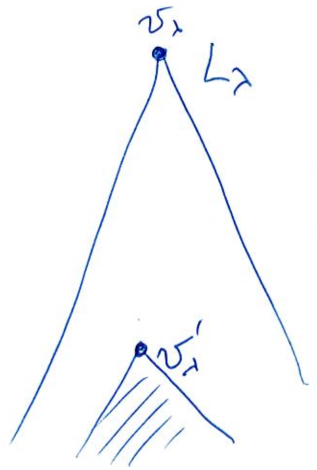
$e_j(v_\lambda') = 0 \quad \forall j$

If $v_\lambda' \neq 0$, then:

the submodule generated by $v_\lambda' \subset L_\lambda$ is gonna be a proper submodule.

$\Downarrow$

$\Downarrow$ contradiction with the irreducibility of L_λ!



So: v_λ must vanish; i.e. $\boxed{v_\lambda' = 0}$

$\Downarrow$

v_λ is of fin. type / $\mathfrak{sl}_2^{(i)}$

L_λ is gen'd by v_λ

Prop 2

L_λ - integrable.

$\checkmark$

■

SO: $\boxed{L_\lambda\text{ - integrable} \iff \lambda(h_i) \in \mathbb{Z}_{\geq 0} \quad \forall i}$

Def: The weight $\lambda \in \mathfrak{h}^*$ is $\boxed{\text{integral dominant}}$, denoted $\lambda \in P_+$, if $\boxed{\lambda(h_i) \in \mathbb{Z}_{\geq 0} \quad \forall i}$

! **Rmk**: If $\mathfrak{g} = \mathfrak{g}(A)$ - simple f. dim, then $\boxed{L_\lambda\text{ - integrable} \iff L_\lambda\text{ - fin. dim.}}$

<u>Goal</u>: Find $ch(L_\lambda)$ for $\lambda \in P_+$.

(generalizing classical Weyl character f-la).

<u>Next time</u>.

Weyl group of Kac-Moody $\mathfrak{g}(A)$

$P = \mathfrak{h}^* \oplus \underbrace{F}_{Q \otimes_{\mathbb{Z}} \mathbb{C}}$, $\quad (\cdot,\cdot): P \times P \to \mathbb{C}$ - symm. pairing (from the beginning of today's class)

Def: For $i \in \{1, \ldots, r\}$, define a linear map "simple reflection"

$$\boxed{r_i : P \to P, \quad \alpha \mapsto \alpha - \alpha(h_i) \cdot \alpha_i}$$

Lemma 1 (Exercise): (a) $r_i^2 = Id$

(b) $(r_i(x), r_i(y)) = (x, y) \quad \forall x, y \in P$.

Lemma 2: If V is an integrable $\mathfrak{g}(A)$-module, then $\forall \mu \in P$ $\exists$ isomorphism $V[\mu] \xrightarrow{\sim} V[r_i(\mu)]$.

$$\Downarrow$$

$$\boxed{\dim(V[\mu]) = \dim(V[r_i(\mu)])}$$

Proof of Lemma 2

▸ V-integrable $\overset{sl_2\text{-theory}}{\Longrightarrow}$ $\mu(h_i)\in\mathbb{Z}$ for any μ s.t. $V[\mu]\neq 0$.

$$(r_i\mu)(h_i)=(\underbrace{\mu-\mu(h_i)\alpha_i}_{r_i(\mu)})(h_i)=\mu(h_i)-2\mu(h_i)=-\mu(h_i)\in\mathbb{Z}$$

One of $\{\mu(h_i),\ \underbrace{(r_i(\mu))(h_i)}_{=-\mu(h_i)}\}$ is in $\mathbb{Z}_{\geq 0}$, WLOG it's μ.

Then, by sl_2-theory:

$$\boxed{V[\mu]\ \underset{e_i^{\mu(h_i)}}{\overset{f_i^{\mu(h_i)}}{\rightleftarrows}}\ V[r_i\mu] \ \text{— isom. of v. spaces}}$$

∎

Def: The *Weyl group* of $\mathfrak{g}(A)$ is the subgroup $W\subseteq GL(P)$ generated by simple reflections $\{r_i\}$

$$\boxed{W=\langle r_i\rangle_{i\in I}}$$

Rmk: $r_i(\alpha_j)=\alpha_j-a_{ij}\cdot\alpha_i \Rightarrow W$ preserves subspace $F\subseteq P$ & W acts by Identity on P/F

$\Rightarrow$ can view $W\subseteq GL(F)$