Lecture #23

04/20/2021

Last time

(1) Casimir Operator $\Delta : M \circlearrowleft$ $\forall$ module $M \in \mathcal{O}$-category

- commutes with $\mathfrak{g}(A)$-action
- acts on h.wt modules with h.wt λ via $(\lambda, \lambda+2\rho)\cdot \mathrm{Id}$.

(2) Integrable Modules

M - $\mathfrak{g}(A)$-mod is integrable if it's loc. finite w.r.t. each $sl_2^{(i)} \subseteq \mathfrak{g}(A)$

(Lie subalg. gen-d by $\{e_i, f_i, h_i\} \simeq sl_2$)

L_λ - integrable $\Longleftrightarrow$ $\lambda \in \underline{P_+} = \{\lambda \in \mathfrak{h}^*_{ext} \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \ \forall i\}$

(3) Weyl Group $\qquad P = \mathfrak{h}^* + F$

complexified root lattice i.e. $F = Q \otimes_{\mathbb{Z}} \mathbb{C}$.

Simple reflections $\boxed{r_i : P \circlearrowleft \quad \alpha \mapsto \alpha - \alpha(h_i)\alpha_i}$, in particular $\boxed{\alpha_j \mapsto \alpha_j - a_{ij}\alpha_i}$

Weyl Group $W \subseteq GL(P)$ is gen-d by $\{r_i\}$

Note: W preserves $F \subseteq P$ and W acts trivially on P/F

$\Rightarrow$ can also view W as subgp of $GL(F)$.

Today: Weyl–Kac character formula.

Thm: Let $\lambda \in P_+$ be a dominant integral weight of Kac–Moody $\mathfrak{g}(A)$.
Let V be an integrable h.wt. module (of $\mathfrak{g}^{ext}(A)$ or $\mathfrak{g}(A)$) with h.wt λ. Then:

(1) $V \simeq L_\lambda$.

(2) $$\boxed{\mathrm{ch}\, V = \sum_{w \in W} \underbrace{\det(w)}_{=\pm 1} \cdot \mathrm{ch}\, M_{w(\lambda+\rho)-\rho} = \sum_{w \in W} \frac{\det(w)\, e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}}$$

Weyl–Kac character formula!

Here: • $w \in W$, $w = r_{i_1} r_{i_2} \dots r_{i_k} \Rightarrow \det(w) = (-1)^k$.

• $\rho \in \mathfrak{h}^* \subseteq P$ – as last time.

Rmks: ① (2) $\Rightarrow$ (1)

Note: h.wt. module V admits epimorphism $V \twoheadrightarrow L_\lambda$

$\lambda \in P_+ \Rightarrow L_\lambda$ - integrable

know: V - integrable

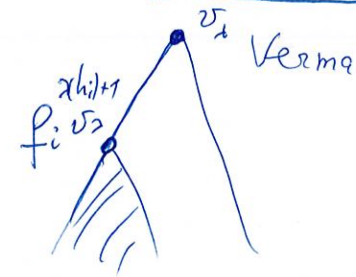
$\overset{(2)}{\Longrightarrow} ch(V) = ch(L_\lambda)$

$\Rightarrow$ $V \xrightarrow{\sim} L_\lambda$.

So: It suffices to prove part (2) of the theorem

② Consider ($\lambda \in P_+$)

$$\boxed{L'_\lambda = M_\lambda \Big/ \big(f_i^{\lambda(h_i)+1}(v_\lambda)\big)}$$



Following proofs from last time; it's easy to see:

L'_λ - integrable. (just b/c L'_λ is generated by the image of v_λ which can be obviously included in f.dim. $sl_2^{(i)}$-submod).

① $\Rightarrow$ $L'_\lambda \simeq L_\lambda$

! Conclusion: L_λ can be explicitly characterized as a quotient of Verma M_λ (for $\lambda \in P_+$)

③ Applying Weyl-Kac char. f-la to $\lambda = 0$, we obtain

$$1 = \sum_{w\in W} \frac{\det(w)\cdot e^{w\rho-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}} \Rightarrow \boxed{\prod_{\alpha>0}(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha} = \sum_{w\in W} \det(w)\, e^{w\rho-\rho}}$$

"Weyl-Kac denominator f-la"

$$\boxed{ch(L_\lambda) = \frac{\sum_{w\in W} \det(w)\cdot e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W} \det(w)\cdot e^{w\rho-\rho}}}$$

another, more symmetric, form of the Weyl-Kac character f-la.

④ For $\mathfrak{g}(A)$ - simple f.d., we recover the classical Weyl character f-la.

For the rest of today, we shall prove part (2) of the theorem!

We'll start from ~10 Lemmas (the proof will easily follow)

Lemma 1: (a) $(\cdot,\cdot)$ is W-inv. $\Leftarrow$ r_i-inv. from last time

(b) V-integrable, $\mu \in P$, $w \in W$ $\Rightarrow$ $\boxed{V[\mu] \simeq V[w(\mu)]}$ $\Rightarrow$ same dim. $\leftarrow$ follows from $w = r_i$ done last time.

(c) Applying (b) to $\mathfrak{g}(A) \overset{ad}{\curvearrowright} \mathfrak{g}(A)$, get W permutes the set of roots and $\boxed{\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{w(\alpha)}).\ \forall w \in W\ \forall \text{root } \alpha}$

! (d) $r_i(\alpha_i) \mapsto -\alpha_i$, r_i permutes the remaining positive roots $\Delta^+ \setminus \{\alpha_i\}$.

(a, b, c) - easy as mentioned above.

(d) $r_i(\alpha_i) = \alpha_i - \underbrace{a_{ii}}_{=2} \cdot \alpha_i = -\alpha_i$.

Pick $\alpha \in \Delta^+ \setminus \{\alpha_i\}$, $\alpha = \sum k_j \alpha_j$ with all $k_j \geq 0$ & at least one of $k_{j \neq i} > 0$.

$r_i(\alpha) = \alpha - \text{coeff} \cdot \alpha_i = \sum_{j \neq i} k_j \alpha_j + (k_i - \ldots) \cdot \alpha_i$. — still a root of $\mathfrak{g}(A)$. $\Rightarrow$

$\Rightarrow r_i(\alpha)$ - positive root.

Lemma 2: Consider $\boxed{K := e^{\rho} \cdot \prod_{\alpha > 0} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}$ $\leftarrow$ $e^{\rho}\cdot$ "denominator in the Weyl-Kac formula"

Then: K is W-anti-invariant, i.e. $\boxed{w(K) = \underbrace{\det(w)}_{\in\{\pm 1\}} \cdot K}$

- Pick i.
- $K = e^{\rho} \cdot (1-e^{-\alpha_i})^1 \cdot \underbrace{\prod_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}_{r_i\text{-invariant due to Lemma 1(c,d)}}$
- $r_i(1-e^{-\alpha_i}) = 1-e^{\alpha_i}$
- $r_i(e^{\rho}) = e^{r_i(\rho)} = e^{\rho - \overset{=1}{\rho(h_i)} \cdot \alpha_i} = e^{\rho} \cdot e^{-\alpha_i}$

$\Rightarrow r_i(K) = -K \Rightarrow$

$\Rightarrow$ result $\forall w$ as W is gen-d by r_i's. ∎

$\boxed{\text{Lemma 3}}$: Let $\lambda \in P_+$ (i.e. integral dominant). Then:

a) $W(\lambda) \subseteq D(\lambda)$, where $D(\lambda) = \{\lambda - \sum_i n_i \alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$ – as before

b) If $D \subseteq D(\lambda)$ is W-inv. set, then $D \cap P_+ \neq \emptyset$.

a) $\lambda \in P_+ \rightsquigarrow$ irred. L_λ - integrable $\xRightarrow{\text{Lemma 1(b)}}$ $\underbrace{P(L_\lambda) := \{\text{weights of } L_\lambda\}}$ is a W-inv. set. $\Rightarrow$

$$\underbrace{w(\lambda) \in P(L_\lambda). \qquad P(L_\lambda) \subseteq D(\lambda).}$$

$\Rightarrow w(\lambda) \in D(\lambda)$. ✓

b) Pick any $\psi \in D \Rightarrow$ choose $w \in W$ s.t. $\lambda - w(\psi)$ is of smallest height.

defined as $\sum_i n_i$ in the expression $\lambda - \sum n_i \alpha_i$

$\boxed{\text{Claim}}$: $w(\psi) \in P_+$!

Assume not $\Rightarrow \exists i$ s.t. $(w\psi, \alpha_i) < 0 \Rightarrow r_i(w\psi) = w\psi + (\text{positive integer}) \cdot \alpha_i$

$\Rightarrow \lambda - r_i w(\psi)$ is of smaller height $\Rightarrow$ ↯

$\square$

Cor 1: $\forall w \in W \setminus \{1\}$ $\exists i$ s.t. $w(\alpha_i) < 0$.

▶ Choose $\chi \in P_+$ s.t. $w\chi \neq \chi$. $\overset{w^{-1}}{\Rightarrow}$ $w^{-1}\chi \neq \chi \Rightarrow \underline{w^{-1}\chi = \chi - \sum k_i \alpha_i}$, $k_i \in \mathbb{Z}_{\geq 0}$.

Apply w : $\underline{\chi} = w\chi - \sum k_i w(\alpha_i) = (\underline{\chi} - \sum \underbrace{k_i'}_{\in \mathbb{Z}_{\geq 0}} \alpha_i) - \sum k_i w(\alpha_i)$

$\Rightarrow \underline{\sum k_i' \alpha_i + \sum k_i w(\alpha_i) = 0}$. $\Longrightarrow$ $w(\alpha_i)$ is negative for some i.

simple positive roots

∎

Lemma 4 Let $\varphi, \psi \in P$ s.t. $\varphi(h_i) > 0$, $\psi(h_i) \geq 0$ $\forall i$. Then
$\underline{w\varphi = \psi \Longleftrightarrow w = Id,\ \varphi = \psi}$.

▶ $\varphi(h_i) > 0 \Longleftrightarrow (\varphi, \alpha_i) > 0$. $\forall i$.

If $w \neq 1$, then by Cor 1 $\exists i$ s.t. $w(\alpha_i) < 0$. Then:

$0 < (\varphi, \alpha_i) = (w^{-1}\psi, \alpha_i) \overset{\text{Lemma 1}}{=\!=} (\psi, \underbrace{w(\alpha_i)}_{\in \Delta^-}) \leq 0 \Rightarrow \lightning \Rightarrow w = 1 \Rightarrow \varphi = \psi$.

∎

Lemma 5: Let $\mu, \nu \in P_+$ be s.t. $\mu \in D(\nu)$, $\mu \neq \nu$.

Then: $(\nu+\rho, \nu+\rho) - (\mu+\rho, \mu+\rho) > 0$.

$\mu \in D(\nu) \Rightarrow \mu = \nu - \sum k_i \alpha_i$, $\quad k_i \in \mathbb{Z}_{\geq 0}$ (& not all zero!)

$$(\nu+\rho, \nu+\rho) - (\mu+\rho, \mu+\rho) = (\nu,\nu) - (\mu,\mu) + (\nu, 2\rho) - (\mu, 2\rho) = (\underbrace{\nu-\mu}_{=\sum k_i\alpha_i}, \overbrace{\nu+\mu+2\rho}^{\in P_{++} \text{ strictly dominant}}) =$$

$$= \sum k_i \underbrace{(\alpha_i, \nu+\mu+2\rho)}_{\in \mathbb{Z}_{>0}} > 0.$$

■

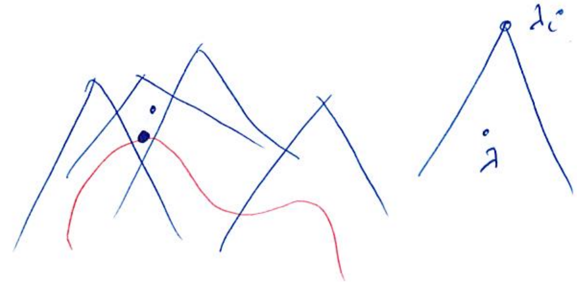
Lemma 6: For any $V \in \mathcal{O}$, we have

$$\boxed{ch(V) = \sum_{\alpha} c_\alpha \cdot ch(M_\alpha), \text{ with } c_\alpha \in \mathbb{Z} \text{ and } \alpha \in \bigcup_{\lambda \in P(V)} D(\lambda)}$$

$\underbrace{P(V)}_{\text{support of the module } V}$ = set of weights of V.

▶ $V \in \mathcal{O} \Rightarrow \exists\, \lambda_1, \dots, \lambda_m \in P$ s.t. $\underbrace{P(V) \subseteq \bigcup_{i=1}^{m} D(\lambda_i)}$

- $\forall \underbrace{\lambda \in D(\lambda_i)}_{\lambda = \lambda_i - \sum_j k_j \alpha_j}$, let $\underbrace{h_i(\lambda)}_{\text{"height of } \lambda \text{ w.r.t. } \lambda_i\text{"}} = \sum k_j$ where



- $\forall \lambda \in \bigcup_{i=1}^{m} D(\lambda_i)$, set $\underbrace{h(\lambda)}_{\text{"combined height"}} = \sum_{i:\, \lambda \in D(\lambda_i)} h_i(\lambda)$

- Finally, set $\boxed{h(V) := \min_{\lambda \in P(V)} h(\lambda)}$ & let $\mu_1, \dots, \mu_r$ - els of $P(V)$ with this "minimal" height.

and let $\{v_{i1}, \dots, v_{i,s_i}\}_{i=1}^{r}$ be the bases of $V[\mu_i]$ $(1 \le i \le r)$.

Then:

$$0 \to \underset{\underset{Ker(\varphi)}{\parallel}}{K} \to \bigoplus M_{\mu_i}^{\oplus s_i} \xrightarrow{\ \varphi\ } V \to \underset{\underset{Coker(\varphi)}{\parallel}}{C} \to 0, \qquad s_i = \dim V[\mu_i]$$

exact sequence of $\mathfrak{g}(A)$-modules

Then: • $ch(V) = \sum \dim V[\mu_i] \cdot ch(M_{\mu_i}) - ch(K) + ch(C)$.

• $P(K),\ P(C) \subseteq \bigcup_{i=1}^{m} D(\lambda_i)$

• $h(K),\ h(C) > h(V)$

$\Rightarrow$ can apply same argument to the modules K, C instead of V

$$\boxed{ch(V) = \sum_{\substack{\text{possibly} \\ \text{infinite sum}}} c_\alpha \cdot ch\, M_\alpha}$$

Note: It will be important how this f-la was derived, not just the final result. In particular, all Verma modules we encounter are equipped with non-trivial $\mathfrak{g}(A)$-homom-s to various subquotients of V!

Proof of Theorem.

- Step 1: by previous Lemma:

$$\boxed{\operatorname{ch}(V) = \sum_{\psi \in D(\lambda)} c_\psi \cdot \operatorname{ch}(M_\psi), \quad c_\lambda = 1.}$$

$\mathbb{Z}$-coeff.

Vermas

h. wt. integr. repr of weight λ

Want: $\operatorname{ch}(V) \overset{?}{=} \sum_w \det(w) \cdot \operatorname{ch}(M_{w(\lambda+\rho)-\rho})$.

- Step 2: Estimate for non-vanishing of c_ψ.

Lemma 7: If $c_\psi \neq 0 \Rightarrow (\psi+\rho, \psi+\rho) = (\lambda+\rho, \lambda+\rho)$.

▶ Look at $\bigoplus M_{\mu_i}^{\oplus \ldots} \longrightarrow V$ – $\mathfrak{g}(A)$-mod. homom.

$\circlearrowleft \Delta \leftarrow$ Casimir $\rightarrow \Delta$ $\circlearrowleft$

Casimir commutes with $\mathfrak{g}(A) \Rightarrow (\lambda, \lambda+2\rho) = (\mu_i, \mu_i+2\rho)$

$$\Rightarrow \underbrace{(\psi, \psi+2\rho)}_{(\psi+\rho,\psi+\rho)-(\rho,\rho)} = \underbrace{(\lambda, \lambda+2\rho)}_{(\lambda+\rho,\lambda+\rho)-(\rho,\rho)} \quad \forall \psi \text{ s.t. } c_\psi \neq 0. \quad \checkmark \quad \blacksquare$$

• Step 3: Detecting terms from Weyl-Kac f-la

Lemma 8: If $\psi = w(\chi+\rho)-\rho$ (so that $\psi+\rho = w(\chi+\rho)$), then

$$\boxed{c_\psi = \det(w)\cdot \underbrace{c_\chi}_{=1} = \det(w).}$$

$$\underbrace{ch(V)}_{\sum c_\psi\, ch(M_\psi)} \cdot \underbrace{K}_{e^\rho \cdot \prod_{\alpha>0}(1-e^{-\alpha})^{\dim g_\alpha}} = \sum_\psi c_\psi \cdot e^{\psi+\rho}$$

Note: • V-integrable $\xRightarrow{\text{Lemma 1}}$ $ch(V)$ — W-inv.
• Lemma 2: K is W-anti-invariant

$\Rightarrow$ $ch(V)\cdot K$ is W-antiinvariant

$$\Rightarrow \boxed{\sum_\psi c_\psi e^{\psi+\rho} \text{ is } W\text{-inv.}}$$

$$w\Big(\sum_\psi c_\psi e^{\psi+\rho}\Big) = \sum c_\psi e^{w(\psi+\rho)}$$

// W-anti-inv.

$$\det(w)\cdot \sum c_\psi e^{\psi+\rho}$$

$$\Rightarrow \boxed{c_{w(\psi+\rho)-\rho} = \det(w)\cdot c_\psi \quad \forall \psi,\ \forall w}$$

• Step 4: End of Proof

Lemma 9: Let $D := \{\psi \mid c_{\psi-\rho} \neq 0\}$

Then: $D = W(\lambda+\rho)$

! This completes the proof of Thm!

▶ By previous Lemma: $W(\lambda+\rho) \subseteq D$. Assume $\underset{W\text{-inv.}}{D} \setminus \underbrace{W(\lambda+\rho)}_{W\text{-inv.}} \neq \emptyset$. Then:

$$\underbrace{\left(D \setminus W(\lambda+\rho)\right) \cap P_+}_{\Downarrow} \neq \emptyset \quad \text{by Lemma 3(b).}$$

Pick such β.

By def-n: $\beta - \rho \in D(\lambda)$

[Actually we need a slight upgrade as we don't know that $\beta-\rho \in P_+$. Nevertheless, following the proof of Lemma 5, get:
$$(\lambda+\rho, \lambda+\rho) - (\beta, \beta) = \sum \underset{\substack{\in \mathbb{Z}_{\geq 0} \\ \text{(not all zero)}}}{k_i} (\alpha_i, \underbrace{\lambda+\beta+\rho}_{\in P_{++}}) > 0$$
$\underbrace{\quad}_{>0 \ \forall i}$]

$$\left.\begin{array}{ll} (\lambda+\rho, \lambda+\rho) - (\beta, \beta) > 0 & \text{(Lemma 5)} \\ (\lambda+\rho, \lambda+\rho) = (\beta, \beta) & \text{(Lemma 7)} \end{array}\right\} \Rightarrow \lightning \Rightarrow \boxed{D = W(\lambda+\rho)}$$

[Note: $W \longrightarrow P$ is 1-to-1 by Lemma 4
$w \longmapsto w(\lambda+\rho) - \rho$]

✓