Lecture #24

04/22/2021

$\boxed{\text{Last time}}$ Kac-Weyl char. f-la.

- $\lambda \in P_+$ ← dominant integral
- V - h.wt. integrable $\mathfrak{g}(A)$-module of h.wt λ.

Then:

$$ch(V) = \sum_{w \in W} \underbrace{\det(w)}_{=\pm 1} \cdot \underbrace{ch\, M_{w(\lambda+\rho)-\rho}}_{\text{character of Verma}} = \sum_{w \in W} \frac{\det(w) \cdot e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}$$

use Kac-Weyl denominator formula

$$= \frac{\sum_{w \in W} \det(w)\, e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} \det(w)\, e^{w\rho-\rho}}.$$

$\boxed{\text{Here: numerator \& denominator are of the same form}}$

$\boxed{\text{Today \& Next time}}$: Apply to $\mathfrak{g}(A) = \widehat{\mathfrak{sl}}_2 \rightsquigarrow \widetilde{\mathfrak{sl}}_2 = \widehat{\mathfrak{sl}}_2 \rtimes \mathbb{C}d$.

! You can find same discussions in [Kac-Raina, § 11-12].

• $\mathfrak{g}$ – simple f.d. Lie alg.

$\rightsquigarrow L\mathfrak{g} = \mathfrak{g}[t,t^{-1}] \rightsquigarrow \hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}\cdot K$ – also it's Kac-Moody

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \rtimes \mathbb{C}d = L\mathfrak{g} \oplus \underbrace{\mathbb{C}\cdot K}_{\text{central}} \oplus \overbrace{\mathbb{C}d}^{\text{derivation}}$$

$\leftarrow$ extended affine Kac-Moody

[Recall: $[d, xt^n] = n\cdot xt^n$
$[d, K] = 0$]

Similar to $\mathfrak{g} = \mathfrak{sl}_n$ (treated in Lecture 15), this extended algebra $\tilde{\mathfrak{g}}$ can be equipped with non-deg. symmetric inv. pairing via:

- $(a(t), b(t)) = \mathrm{Res}_{t=0}\, (a(t), b(t)) \cdot \frac{dt}{t}$

  [i.e. $(x\cdot t^n, y\cdot t^m) = (x,y)\cdot n\,\delta_{n,-m}$
  $\forall n,m \in \mathbb{Z}$
  $\forall x,y \in \mathfrak{g}$]

- $(K,d) = 1, \quad (K,K) = 0 = (d,d) = (K, L\mathfrak{g}) = (d, L\mathfrak{g})$.

$\boxed{\text{Lemma 1: Exercise}}$ (a) Prove the above f-las endow $\tilde{\mathfrak{g}}$ with a nondeg. inv. pairing!

(b) It is the unique, up to scalar, once you impose $(d,d)=0$.

• Recall: For simple f.d. $\mathfrak{g}$, the Weyl group can be equivalently described:

$$\boxed{W = N(T)/T}$$

← alternative description of finite Weyl gps

[$T \subseteq G$ - max torus, $\mathrm{Lie}\, G = \mathfrak{g}$
$N(T)$ = normalizer of T in G]

• $\mathfrak{g}$ - simple f.d.
G - simply-connected gp with $\mathrm{Lie}(G) = \mathfrak{g}$.

$G \subseteq \mathrm{Mat}_{N\times N}(\mathbb{C})$ 'algebraic, i.e. determined by several pol-l relations.

$G \overset{Ad}{\curvearrowright} \mathfrak{g}$ ⇝ $\boxed{\underline{\underline{LG}} \curvearrowright L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]}$

Define $LG \subseteq \mathrm{Mat}_{N\times N}(\mathbb{C}[t, t^{-1}])$ by the same rel-s

loop group

Lemma 2 : The action $LG \curvearrowright L\mathfrak{g}$ can be uniquely extended, to $LG \curvearrowright L\mathfrak{g} \oplus \mathbb{C}d$.

i. e. compatible w.r.t. embedding $L\mathfrak{g} \hookrightarrow L\mathfrak{g} \oplus \mathbb{C}d$.

Exercise

via

$$\boxed{LG \ni g(t) : d \longmapsto d - t \cdot g'(t) \cdot g(t)^{-1}}$$

Prop 1 : There is an action of LG on $\tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C} \cdot K \oplus \mathbb{C}d$ given by:

Exercise

$$\boxed{\begin{aligned} g_{(t)}(K) &= K \\ g_{(t)}(\underset{L\mathfrak{g}}{\overset{\cap}{a}}) &= gag^{-1} + \operatorname{Res}_{t=0} tr(g'ag^{-1})dt \cdot K \\ g_{(t)}(d) &= d - tg'(t)g(t) - \tfrac{1}{2} \operatorname{Res}_{t=0} (tg'g^{-1}, tg'g^{-1})\tfrac{dt}{t} K \end{aligned}}$$

it is a unique action compatible with that in Lemma 2!

$$\left[\begin{array}{c} \tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}d \oplus \mathbb{C}K \\ \big\downarrow {\scriptstyle K \mapsto 0} \\ L\mathfrak{g} \oplus \mathbb{C}d \end{array}\right]$$

$\boxed{\text{Take away}}$ $\quad G \overset{Ad}{\curvearrowright} \mathfrak{g} \rightsquigarrow \boxed{LG \curvearrowright \tilde{\mathfrak{g}}}$!

• From now on, we shall focus on the simplest case of $\mathfrak{g}$:

$$\boxed{\mathfrak{g} = \mathfrak{sl}_2} \qquad \mathfrak{sl}_2 = \langle e, h, f \rangle$$

(Recall Notations of Lecture 15)

Let $\tilde{\mathfrak{h}} \subseteq \tilde{\mathfrak{g}}$ - Cartan of the extended Kac-Moody

basis: $h, K, d \rightsquigarrow \underline{h_0 = K - \alpha, \; h_1 = \alpha}$, (see Lecture 15)

($h \leftrightarrow \alpha$ via pairing)

$$\boxed{\begin{array}{l} \text{Pairing } \tilde{\mathfrak{h}} \times \tilde{\mathfrak{h}} \longrightarrow \mathbb{C} \\ (\alpha, \alpha) = 2, \; (K, d) = 1 = (d, K), \; \text{others} = 0. \end{array}}$$

Fundamental Weights: $\boxed{\begin{cases} \omega_0 = d \\ \omega_1 = \frac{1}{2}\alpha + d \end{cases}}$ i.e. $\begin{array}{l} (\omega_0, h_1) = 0, \; (\omega_0, h_0) = 1, \; (\omega_0, d) = 0 \\ (\omega_1, h_1) = 1, \; (\omega_1, h_0) = 0, \; (\omega_1, d) = 0. \end{array}$

Any weight can be written as $\boxed{\lambda = m \cdot d + \frac{n}{2} \cdot \alpha + r \cdot K = (m-n)\,\omega_0 + n \cdot \omega_1 + r \cdot K}$

[elements of $\tilde{\mathfrak{h}}^*$, identified with $\tilde{\mathfrak{h}}$ via the above pairing]

- Lecture 15 : L_λ - unitary $\iff$ $m \geq n \in \mathbb{N}$, $\tau \in \mathbb{R}$.

- Lectures 22-23 : L_λ - integrable $\iff$ $m \geq n \in \mathbb{N} = \mathbb{Z}_{\geq 0}$.

- The level of L_λ equals m

Def (Alternative description of the Weyl group)

set $\widetilde{W} := \{ g \in LG \mid g \tilde{\mathfrak{h}} g^{-1} \subseteq \tilde{\mathfrak{h}} \}$ (stabilizes the extended Cartan)

Weyl group $\rightarrow$ $W := \mathrm{Im} \left(\widetilde{W} \to \mathrm{End}(\tilde{\mathfrak{h}}) \right)$ - subgroup of $\mathrm{End}(\tilde{\mathfrak{h}})$

! Fact: (non-trivial) This definition coincides with the previous, where the Weyl gp was defined as a subgp of $\mathrm{End}(\tilde{\mathfrak{h}})$ generated by simple reflections

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• Let's compute $\widetilde{W}$ explicitly in the case of $\mathfrak{sl}_2$.

Recall (from Prop 1): $g(t)(\alpha) = \underbrace{g\alpha g^{-1}}_{\in L\mathfrak{sl}_2} + \underbrace{\mathrm{Res}_{t=0}\, \mathrm{tr}(g'(t)\alpha g(t)^{-1})dt}_{\text{scalar}} \cdot K.$ $\qquad \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\qquad \mathfrak{h} \subset \widetilde{\mathfrak{h}}$

So: $g(t) \in \widetilde{W} \Rightarrow g(t) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g(t)^{-1} = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix}$ for some $\nu \in \mathbb{C}$.

But: $\det(\text{LHS}) = -1 \Rightarrow -\nu^2 = -1 \Rightarrow \nu = \pm 1$.

$\boxed{\text{Case 1}}$ ($\nu = 1$): $g(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g(t)$. Find $g(t)$!

$$\Downarrow$$

$g(t) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $\det g(t) = 1 \Rightarrow g(t) = \begin{pmatrix} a(t) & 0 \\ 0 & a(t)^{-1} \end{pmatrix}$

$\quad \cap$

$\quad LG$

But: $a(t), \frac{1}{a(t)}$ – Laurent pol-s.

$$\Downarrow$$

$\boxed{a(t) = c \cdot t^k \text{ for some } k \in \mathbb{Z},\ c \in \mathbb{C}^\times}$

Conclusion: $g(t) = \begin{pmatrix} ct^k & 0 \\ 0 & c^{-1}t^{-k} \end{pmatrix}$

$\boxed{\text{Case 2}}$ ($\nu = -1$): $g(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g(t)$.

$$\Downarrow$$

$g(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a(t) & 0 \\ 0 & a(t)^{-1} \end{pmatrix}$

But: $a(t), \frac{1}{a(t)}$ – Laurent pol-s

$$\Downarrow$$

$\boxed{a(t) = c \cdot t^k,\ k \in \mathbb{Z},\ c \in \mathbb{C}^\times}$

Conclusion: $g(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} ct^k & 0 \\ 0 & c^{-1}t^{-k} \end{pmatrix}$

<u>Easy</u>: El-s $g(t)$ from both Cases 1 & 2 map $\mathfrak{d}$ to el-s $\mathfrak{h}$.

<u>So</u>: $\boxed{\widetilde{W} \text{ is generated by } \left\{ \begin{pmatrix} ct^k & 0 \\ 0 & c^{-1}t^{-k} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} ct^k & 0 \\ 0 & c^{-1}t^{-k} \end{pmatrix} \right\}_{c \in \mathbb{C}^\times}^{k \in \mathbb{Z}}}$

$$W = \text{image of } \widetilde{W} \longrightarrow \operatorname{End}(\mathfrak{h})$$

<u>Define</u>:

$W \ni t_k$ is the image of $\begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} \in \widetilde{W}$

$W \ni r_\alpha$ is — " — $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \widetilde{W}$

<u>Exercise</u>: $r_\alpha : \alpha \mapsto -\alpha,\ K \mapsto K,\ d \mapsto d$

$t_k : \alpha \mapsto \alpha + 2k \cdot K,\ K \mapsto K,\ d \mapsto d - k \cdot \alpha - k^2 \cdot K$

<u>Outcome</u>: The Weyl group of $\hat{\mathfrak{sl}}_2$ consists of $\{t_k, r_\alpha t_k\}$ whose action is explicitly given by the f-las above

$\boxed{\text{Remark}}$: Note that $W = \{t_k, r_\alpha t_k\}_{k\in\mathbb{Z}} = \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\substack{\text{gen.d} \\ \text{by } r_\alpha}} \ltimes \underbrace{\mathbb{Z}}_{\substack{\text{consisting} \\ \text{of } t_k}}$

This is a particular case of the general result:

$$\boxed{\underbrace{W^{aff}}_{\substack{\text{affine} \\ \text{Weyl gp}}} \simeq \underbrace{W^{fin}}_{\substack{\text{finite} \\ \text{Weyl gp}}} \ltimes \underbrace{Q^\vee}_{\substack{\text{coroot} \\ \text{lattice}}} \simeq \mathbb{Z}^r \quad \text{with } r = \mathrm{rk}(\mathfrak{g})}$$

Our key Objective is to write down the Weyl-Kac char. f-la for $\mathfrak{sl}_2$ using the above description of the Weyl gp of $\mathfrak{sl}_2$.

<u>Weyl-Kac character formula</u> : $\mathrm{ch}_{L_\lambda} = \dfrac{\sum_{w\in W} \det(w)\, e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W} \det(w)\, e^{w(\rho)-\rho}}$

For $h \in \mathfrak{h}$, we can evaluate $\mathrm{ch}_{L_\lambda}(h)$ which essentially computes $\mathrm{Tr}_{L_\lambda}(e^h)$.

Warning : If L_λ - ∞-dim, need to be careful with the trace above.

[We'll apply this to the general elt of $\mathfrak{h}^*$ and view the trace as a converging function.]

So :

$$\boxed{\mathrm{ch}_{L_\lambda}(h) = \frac{\sum_{w\in W} \det(w)\, e^{(w(\lambda+\rho),\, h)}}{\sum_{w\in W} \det(w)\, e^{(w\rho,\, h)}}} \quad (*)$$

← we use the pairing on $\mathfrak{h}$

Take a general elt of $\mathfrak{h}$:

$$\boxed{h = 2\pi i \left(\tfrac{1}{2} z \alpha - \tau \cdot d + u K \right)}$$

- $z, \tau, u \in \mathbb{C}$ – varying parameters
- α, d, K – basis of $\mathfrak{h}$

Def: For $n, m \in \mathbb{Z}$ ($m \neq 0$), define the theta-function:

$$\boxed{\Theta_{n,m}(\tau, z, u) := e^{2\pi i m u} \cdot \sum_{k \in \frac{n}{2m} + \mathbb{Z}} e^{2\pi i m (k^2 \tau + k z)}}$$

Note: absolutely converges for $Im(\tau) > 0$.

In the Kac – Weyl ch. f-la both numerator & denominator are of the form:

$$\boxed{\sum_{w \in W} \det(w)\, e^{(w(\mu), h)}}$$

(either for $\mu = \lambda + \rho$ or $\mu = \rho$)

So, let's compute the sum above explicitly!

$$\boxed{W = \{t_k,\ r_\alpha \cdot t_k\}}$$

Write $\underline{\mu = m\cdot d + \frac{n}{2}\cdot\alpha + r\cdot K}$.

- $\underline{\text{Case of } t_k}$

$\det(t_k) = 1$. Using the f-las of page 8, we get:

$$t_k(\mu) = m(d - k\alpha - k^2K) + \frac{n}{2}(\alpha + 2k\cdot K) + r\cdot K = \underline{m\cdot d + \left(\frac{n}{2} - mk\right)\alpha + (r + kn - k^2 m)K}$$

$$\boxed{(t_k(\mu), \underbrace{h}_{2\pi i\left(\frac{1}{2}z\alpha - \tau\cdot d + u\cdot K\right)}) = 2\pi i\left(z\left(\frac{n}{2} - mk\right) + m\cdot u - \tau(r + kn - k^2 m)\right)}$$

- $\underline{\text{Case of } r_\alpha t_k}$

$\det(r_\alpha t_k) = -1$.

$$r_\alpha t_k(\mu) = md - \left(\frac{n}{2} - mk\right)\alpha + (r + kn - k^2 m)K.$$

$$\boxed{(r_\alpha t_k(\mu), h) = 2\pi i\left(-z\left(\frac{n}{2} - mk\right) + m\cdot u - \tau(r + kn - k^2 m)\right)}$$

Thus :

$$\sum_{w\in \overline{W}} \det(w)\, e^{(w(\mu),h)} = \sum_{k\in\mathbb{Z}} e^{2\pi i m u}\left(e^{2\pi i\left(z\left(\frac{n}{2}-mk\right)-\tau\left(r+kn-k^2m\right)\right)} - e^{2\pi i\left(-z\left(\frac{n}{2}-mk\right)-\tau\left(r+kn-k^2m\right)\right)}\right) =$$

[Let's change k $\longrightarrow$ in the first sum ("+" sign) replace $\underset{\mathbb{Z}}{\underset{\cap}{k}} \rightsquigarrow \underset{\frac{n}{2m}+\mathbb{Z}}{\underset{\cap}{\frac{n}{2m}-k}}$.

in the second sum ("−" sign) replace $\underset{\mathbb{Z}}{\underset{\cap}{k}} \longmapsto \underset{-\frac{n}{2m}+\mathbb{Z}}{\underset{\cap}{\frac{n}{2m}+k}}$]

$q=e^{2\pi i\tau}$

$$= \left(e^{2\pi i\tau\left(-\left(r+\frac{n^2}{4m}\right)\right)} \cdot \Theta_{n,m}(\tau,z,u) - e^{2\pi i\tau\left(-r-\frac{n^2}{4m}\right)} \cdot \Theta_{-n,m}(\tau,z,u)\right) = \boxed{q^{-r-\frac{n^2}{4m}}\cdot \Theta_{n,m}(\tau,z,u) - q^{-r-\frac{n^2}{4m}}\, \Theta_{-n,m}(\tau,z,u)}$$

f-la for $\sum_{w\in W} \det(w)\, e^{(w(\mu),h)}$

for $\mu = m\delta + \frac{n}{2}\alpha + r k$

Conclusion: The Weyl-Kac character f-la for sl_2 becomes:

see eq-n (*) on p. 10

$$ch_{L_\lambda}(h) = q^{-s_\lambda} \cdot \frac{\Theta_{n+1,m+2}(\tau,z,u) - \Theta_{-n-1,m+2}(\tau,z,u)}{\Theta_{1,2}(\tau,z,u) - \Theta_{-1,2}(\tau,z,u)}$$

with $q = e^{2\pi i \tau}$

$s_\lambda = r + \frac{(n+1)^2}{4(m+2)} - \frac{1}{8}$

Here:

$$h = 2\pi i \left(\tfrac{1}{2} z\alpha - \tau \cdot d + u \cdot K\right)$$

$$\lambda = m d + \tfrac{n}{2}\alpha + r K = (m-n)\omega_0 + n\omega_1 + r \cdot K.$$

Note: $\rho = \omega_0 + \omega_1 = 2d + \frac{1}{2}\alpha \longleftrightarrow$ corresponds to $m=2$, $n=1$.