Lecture 25

04/27/2021

$\boxed{\text{Last time}}$ • Weyl–Kac char. f-la for $\widehat{sl}_2$.

Notations: • $\Theta_{n,m}(\tau, z, u) = e^{2\pi i m u} \sum_{k \in \frac{n}{2m} + \mathbb{Z}} e^{2\pi i m (k^2\tau + kz)}$

• $q = e^{2\pi i \tau}$

We ended last lecture with the following key formula (Weyl–Kac for $\widehat{sl}_2$)

$$\boxed{\mathrm{ch}_{L(\lambda)}(h) = q^{-s_\lambda} \cdot \frac{\Theta_{n+1, m+2}(\tau, z, u) - \Theta_{-n-1, m+2}(\tau, z, u)}{\Theta_{1,2}(\tau, z, u) - \Theta_{-1,2}(\tau, z, u)}}$$

$\lambda = m d + \frac{n}{2}\alpha + r K$

$s_\lambda = r + \frac{(n+1)^2}{4(m+2)} - \frac{1}{8}$

$h = 2\pi i \left(\frac{1}{2} z \alpha - \tau \cdot d + u \cdot K\right)$

Cor 1 $\boxed{ch_{L_d}(h) = q^{1/24}\cdot \dfrac{\Theta_{1,3}(\dots) - \Theta_{-1,3}(\dots)}{\Theta_{1,2}(\dots) - \Theta_{-1,2}(\dots)}}$ (1)

▶ Apply the f-la from p.1 to $n=0, m=1$. ∎

Lemma 1: $\boxed{ch_{L_d}(h) = \dfrac{\Theta_{0,1}(\tau, z, u)}{\varphi(q)}, \quad \varphi(q) = \prod_{n>0}(1-q^n)}$ (2)

▶
$$(1) = (2) \iff \Theta_{0,1}\left(\Theta_{1,2} - \Theta_{-1,2}\right) \overset{?}{=} q^{1/24}\cdot\varphi(q)\cdot\left(\Theta_{1,3} - \Theta_{-1,3}\right)$$

[Hwk 13, #4] $\Rightarrow$ (a) $\Theta_{0,1}\cdot\Theta_{1,2} = \Theta_{1,3}\cdot\sum_{k\in\mathbb{Z}-\frac{1}{12}} q^{6k^2} + \Theta_{3,3}\cdot\sum_{k\in\mathbb{Z}+1/4} q^{6k^2} + \Theta_{5,3}\cdot\sum_{k\in\mathbb{Z}+\frac{7}{12}} q^{6k^2}$

(b) $\Theta_{0,1}\cdot\Theta_{-1,2} = \Theta_{-1,3}\cdot\sum_{k\in\frac{1}{12}+\mathbb{Z}} q^{6k^2} + \Theta_{1,3}\cdot\sum_{k\in\mathbb{Z}+\frac{5}{12}} q^{6k^2} + \Theta_{3,3}\cdot\sum_{k\in\mathbb{Z}+3/4} q^{6k^2}$ $\Rightarrow$

! **Note**: $\Theta_{5,3} = \Theta_{-1,3}$, $\sum_{k\in\mathbb{Z}+\lambda} q^{6k^2} = \sum_{k\in\mathbb{Z}-\lambda} q^{6k^2}$

$$\Rightarrow \Theta_{0,1}\left(\Theta_{1,2} - \Theta_{-1,2}\right) = \left(\Theta_{1,3} - \Theta_{-1,3}\right)\cdot\underbrace{\left(\sum_{k\in\mathbb{Z}-\frac{1}{12}} q^{6k^2} - \sum_{k\in\mathbb{Z}+5/12} q^{6k^2}\right)}_{\overset{?}{=}\, q^{1/24}\cdot\varphi(q)}$$

Follows from Euler's pentagonal identity $\Longrightarrow$

$$\varphi(q) = \sum_{m\in\mathbb{Z}} (-1)^m q^{\frac{3m^2+m}{2}}$$

Exercise: Work out the details! ∎

$$\boxed{\mathrm{ch}_{L_\delta}(h) = \frac{\Theta_{0,1}}{\varphi(q)}}$$

$\boxed{\text{Prop 1}}$ [Hwk 13, Problem 4(b)]:

For $\lambda = m\delta + \frac{n}{2}\alpha$ $(m \geq n \geq 0)$, we have:

$$\underbrace{\mathrm{ch}_{L_\delta}(h) \cdot \mathrm{ch}_{L_\lambda}(h)}_{= \mathrm{ch}_{L_\delta \otimes L_\lambda}(h)} = \sum_{k \in I} \psi_{m,n,k}(q) \cdot \mathrm{ch}_{L_{\lambda+\delta-k\alpha}}(h)$$

$$I = \{k \in \mathbb{Z} \mid -\tfrac{1}{2}(m-n+1) \leq k \leq \tfrac{n}{2}\}$$

$$\psi_{m,n,k}(q) = \frac{f_k^{(m,n)}(q) - f_{n+1-k}^{(m,n)}(q)}{\varphi(q)}, \quad f_k^{(m,n)}(q) = \sum_{j \in \mathbb{Z}} q^{(m+2)(m+3)j^2 + [(n+1)+2k(m+2)]j + k^2}$$

(this is a tedious straightforward computation!)

Let's reinterpret this equality from the representation theoretical point of view.

$L_\alpha \otimes L_\lambda \leftarrow$ unitary $\Rightarrow$ semisimple $\Rightarrow$

L_α, L_λ unitary

$$(*) \quad \boxed{L_\alpha \otimes L_\lambda = \bigoplus_{\mu \in P_+} \underbrace{L_\mu}_{\substack{\text{integrable} \\ \text{unitary}}}^{m_\mu} \quad \text{for some } m_\mu \in \mathbb{Z}_{\geq 0}}$$

Prop 1 $\Rightarrow$ explicit f-las for multiplicities m_μ.

$$(*) \Rightarrow \quad ch_{L_\alpha \otimes L_\lambda}(h) = \sum_{\mu \in P_+} m_\mu \cdot ch_{L_\mu}(h) \overset{\text{Prop 1}}{=}$$

$$= \sum_{k \in I} \sum_{j \in \mathbb{Z}} \underbrace{\Delta^j_{m,n,k}}_{\text{coeff-s of } \psi_{m,n,k}(q), \text{ i.e.}} \cdot ch_{L_{\alpha+\lambda-k\alpha-jK}}(h)$$

$$\boxed{\psi_{m,n,k}(q) = \sum_{j \in \mathbb{Z}} \Delta^j_{m,n,k} \cdot q^j}$$

Rmk: The shift by $-jK$ in the h.wt. and appearence of Δ^j is due to the following obvious equality:

$$\boxed{ch_{L_{\mu-jK}}(h) = ch_{L_\mu}(h) \cdot q^j.}$$

Cor 2 $\boxed{L_\alpha \otimes L_\lambda \cong \bigoplus_{\substack{k\in I \\ j\in\mathbb{Z}}} L_{\alpha+\lambda-k\alpha-jK}^{\Delta^j_{m,n,k}}}$

as $\mathfrak{sl}_2$-modules

- Let's now determine the $\min\{j \mid \Delta^j_{m,n,k} > 0\}$ $\forall m,n,k$.

Lemma 2: Define $r, s \in \mathbb{Z}$ via

Exercise

$$\boxed{r = \begin{cases} n+1, & \text{if } k \geq 0 \\ m-n+1, & \text{if } k<0 \end{cases}, \quad s := \begin{cases} n+1-2k, & \text{if } k \geq 0 \\ m-n+2+2k, & \text{if } k<0. \end{cases}}$$

(so that: $1 \leq s \leq r \leq m+1$ $\forall k \in I$ $\forall m \geq n \geq 0$)

Then:

$$\varphi(q)\cdot q^{-k^2}\cdot \psi_{m,n,k}(q) = A+B+C,$$

$$A = 1 - q^{rs} - q^{(m+2-r)(m+3-s)}$$

$$B = \sum_{j>0} q^{(m+2)(m+3)j^2+[(m+3)r-(m+2)s]j}\left(1-q^{2(m+2)sj+rs}\right)$$

$$C = \sum_{j>0} q^{(m+2)(m+3)j^2-[(m+3)r-(m+2)s]j}\left(1-q^{2(m+2)(m+3-s)j+(m+2-r)(m+3-s)}\right)$$

Lemma 3: $\forall m,n,k$ as above, we have

$$\boxed{\min\{j \mid \Delta^j_{m,n,k} \neq 0\} = k^2} \qquad (3)$$

and the corresp. coeff. of q:

$$\boxed{\Delta^{k^2}_{m,n,k} = 1} \qquad (4)$$

$$\psi_{m,n,k}(q) = q^{k^2} \cdot \frac{A+B+C}{\varphi(q)} = q^{k^2} \cdot \underbrace{\frac{1}{\varphi(q)}}_{1+q+\dots} \cdot \underbrace{(A+B+C)}_{(1-q^{rs}-q^{(m+2-r)(m+3-s)} + \text{h.o.t. in } q)}$$

✓

Outcome: f-las (3), (4) which we will use from now on!

Goal 1 Establish unitarity of exceptional series $L_{\cdot,\cdot}$ for Virasoro.

- $L_d, L_\lambda \rightsquigarrow L_d \otimes L_\lambda$

Def: Let $U^{(j)}_{m,n,k} = \{\widehat{sl}_2$ - h.wt. vectors in $L_d \otimes L_\lambda$ of h.wt $= d+\lambda-k\alpha - j\cdot K\}$

Note: $\boxed{\dim U^{(j)}_{m,n,k} = \Delta^j}$!

Warning: Note the difference b/w $\widehat{sl}_2$ & sl_2 !

$U_{m,n,k} = \bigoplus_{j\in\mathbb{Z}} U^{(j)}_{m,n,k} = \{sl_2$ - h. weight vectors in $L_d \otimes L_\lambda$ of h.wt $d+\lambda-k\alpha\}$

Note: $\boxed{tr_{U_{m,n,k}}(q^{-d}) = \psi_{m,n,k}(q)}$

Sugawara Construction $\rightsquigarrow$ "Coset construction" (Goddard-Kent-Olive construction — LECTURE 12)

Outcome: Vir $\curvearrowright L_d \otimes L_\lambda$ commuting with sl_2-action.

$\Downarrow$

$\boxed{\text{Vir} \curvearrowright U_{m,n,k}}$ $\leftarrow$ h.wt. vectors of given weight.

As computed long time ago (in Lecture 18)

$$\boxed{c = 1 - \frac{6}{(m+2)(m+3)}}$$

Let's now compute the action of $L_0 \curvearrowright U_{m,n,k}$.

$$[\text{Hwk } 12,\ \#2(b)]: \quad \boxed{\underbrace{\Delta}_{\text{Casimir Operator}} = 2(\underbrace{k}_{\text{level}} + \underbrace{h^\vee}_{= 2 \text{ for } sl_2})(L_0 + d)}$$

Note: $\Delta|_{L_\mu} = (\mu, \mu + 2\rho)\cdot \mathrm{Id}_{L_\mu}$

Let's now apply coset construction to $L_d \otimes L_\lambda \rightsquigarrow$ L_0 acts on $L_d \otimes L_\lambda$ via:

$$L_0 = \left(\frac{(d, d+2\rho)}{2(1+2)} - d\right) \otimes 1 + 1 \otimes \left(\frac{(\lambda, \lambda+2\rho)}{2(m+2)} - d\right) \otimes 1 - \left(\frac{\Delta|_{L_d \otimes L_\lambda}}{2(m+3)} - d \otimes 1 - 1 \otimes d\right)$$

$$\frac{\lambda = md + \frac{n}{2}\alpha}{\rho = 2d + \frac{1}{2}\alpha} \quad \frac{n(n+2)}{4(m+2)} - \frac{\Delta}{2(m+3)}, \quad \Delta = \Delta|_{L_d \otimes L_\lambda}$$

$$\boxed{L_0 = \frac{n(n+2)}{4(m+2)} - \frac{1}{2(m+3)} \underbrace{\Delta}_{\text{Casimir } \Delta|_{\underline{L_? \otimes L_?}}}}$$

of level $= m+3$

$\longleftarrow$ Conclusion from p.8

Let's now evaluate Δ!

$$\Delta = \underbrace{\tilde{\Delta}_0}_{\text{dual basis of extended Cartan}} + \Delta_+$$

<u>Note</u>: $\tilde{\Delta}_0 = \frac{\alpha^2}{2} + 2Kd$

Let's honestly compute the action of Casimir on each irred. summand L_μ:

$$\Delta = 2(m+3)\cdot d + \frac{\alpha^2}{2} + \overbrace{\underbrace{ef}_{=fe+\alpha} + fe}^{\text{in } sl_2} + \sum_{\substack{a\in B = \text{basis of } sl_2 \\ m>0}} a_{-m}\, a_m.$$

<u>BUT</u>: For any $v \in U_{m,n,k}$: $e(v)=0$, $a_m(v)=0$ $\forall a\in B\ \forall m>0$

$\Longrightarrow$

$$\boxed{\Delta(v) = \left[2(m+3)d + \frac{\alpha^2}{2} + \alpha\right] v \quad \forall v \in U_{m,n,k}}$$

$$\boxed{\Delta(v) = \left[2(m+3)d + \frac{\alpha^2}{2} + \alpha\right](v)}$$ $\leftarrow$ end fla of p. 9.

$$wt(v) = d + \lambda - k\alpha = (m+1)d + \frac{n-2k}{2}\alpha \implies \left(\frac{\alpha^2}{2} + \alpha\right)(v) = \left[\frac{(n-2k)^2}{2} + (n-2k)\right]\cdot v$$

$$\Downarrow$$

$$\boxed{\Delta(v) = \left[2(m+3)d + \frac{(n-2k)^2}{2} + (n-2k)\right] v}$$

Recall: $\boxed{L_0 = \frac{n(n+2)}{4(m+2)} - \frac{1}{2(m+3)}\Delta}$ $\implies$

(top of p. 9)

$$\implies \boxed{L_0\Big|_{U_{m,n,k}} = \frac{n(n+2)}{4(m+2)} - \underbrace{d}_{\text{non-constant operator}} - \frac{(n-2k)(n-2k+2)}{4(m+3)}}$$

BUT: $tr_{U_{m,n,k}}(q^{-d}) = \psi_{m,n,k}(q) \implies -d\big|_{U_{m,n,k}}$ has min. eigenvalue $= k^2$!

$\boxed{\text{Cor 3}}$: The minimal eigenvalue of $L_0 \curvearrowright U_{m,n,k}$ is

$$\boxed{h = \underbrace{k^2 + \frac{n(n+2)}{4(m+2)} - \frac{(n-2k)(n-2k+2)}{4(m+3)}}_{\Vert} \quad \boxed{\frac{[(m+3)r - (m+2)s]^2 - 1}{4(m+2)(m+3)}} \longleftarrow \boxed{h_{r,s}(m)}}$$

THUS :

L_1, L_2 – unitary integrable $/ sl_2$ $\rightsquigarrow$ $L_1 \otimes L_2$ – unitary integr $/ sl_2$ $\longrightarrow$ $Vir \curvearrowright L_1 \otimes L_2$ – unitary

$\cup$

$Vir \curvearrowright \underbrace{U_{m,n,k}}$ (!) semisimple

contains irreducible from exceptional series:

$$\boxed{L_{\underbrace{c(m)}_{1 - \frac{6}{(m+2)(m+3)}},\ h_{r,s}(m)}}$$

Conclusion : Exceptional series for Virasoro are unitary!