04/29/2021

Lecture 26

<u>Last time</u>: Established unitarity of the <u>exceptional series</u> for Virasoro

$$\boxed{L_{\underbrace{h_{r,s}(m)}_{\frac{[(m+3)r-(m+2)s]^2-1}{4(m+2)(m+3)}},\ \underbrace{c(m)}_{1-\frac{6}{(m+2)(m+3)}}}}$$

Outline of the argument

$\mathfrak{sl}_2 \curvearrowright \underbrace{L_d, L_\lambda}_{\text{unitary integrable}} \rightsquigarrow \mathrm{Vir} \curvearrowright L_d \otimes L_\lambda \curvearrowleft \mathfrak{sl}_2 \rightsquigarrow \mathrm{Vir} \curvearrowright$ h.wt. vectors of given weight $\underbrace{U_{m,n,k}}_{\text{unitary}}$

$\boxed{\text{above result}} \Longleftarrow L_{h_{r,s}(m),\, c(m)} \subset U_{m,n,k}$

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Recall the Kac / Feigin - Fuchs det. f-la:

Lecture 14:
$$\boxed{\underbrace{\det_m(c,h)}_{} = \underbrace{\overset{\neq 0}{\text{const}}} \cdot \prod_{\substack{r,s\geq 1 \\ rs\leq m}} \left(h - h_{r,s}(c)\right)^{p(m-rs)}}$$

det of contravariant form (see Lecture 6) on Verma module $M_{c,h}$ of Virasoro in depth m.

Missing part from the proof was:

Thm: $\det_{m=rs}(c,h) = 0$ has a zero at $h = h_{r,s}(c)$.

$$\boxed{\det_{rs}(c, h_{r,s}(c)) = 0}$$

We shall prove it now!

It suffices to show that

$$\boxed{\det_{r,s}(c(m), h_{r,s}(m)) = 0 \qquad \forall\, m \geq r,s}$$

[$r \neq s$: $(h - h_{r,s}(c))(h - h_{s,r}(c))$ – irreducible pol-l.
$r = s$: $h - h_{r,r}(c)$]

$\Updownarrow$

Verma

Irreducible

$$\boxed{M_{c(m), h_{r,s}(m)} \twoheadrightarrow L_{c(m), h_{r,s}(m)}}$$

[Suffices to show that kernel has a nontrivial graded compound in degrees $\leq r \cdot s$.]

$$\mathrm{ch}\, M_{\cdot,\cdot} = \frac{q^{h_{r,s}(m)}}{\varphi(q)} \quad , \varphi(q) = \prod_{n>0} (1-q^n)$$

$$\mathrm{ch}\, L_{c(m),h_{r,s}(m)} \leq \mathrm{ch}\, U_{m,n,k} = \psi_{m,n,k}(q) = \frac{q^{h_{r,s}(m)}}{\varphi(q)} \Big(1 - \underline{\underline{q^{rs}}} - q^{(m+2-r)(m+3-s)} + B + C \Big)$$

$\Downarrow$

implies the result about Ker $(M_{\cdot,\cdot} \twoheadrightarrow L_{\cdot,\cdot})$

This completes finally the proof of the determinant f-la for Virasoro!

$\boxed{\text{Today}}$: [Jantzen] – [Kac–Kazhdan] – [Shapovalov]
determinant formula

the most general setup

$\mathfrak{g} = \mathfrak{g}(A)$ – Kac-Moody alg.

- $\sigma : \mathfrak{g}(A) \circlearrowleft$ – involutive anti-autom. given

$e_i \longmapsto f_i$, $f_i \longmapsto e_i$, $h_i \longmapsto h_i$.

$\Downarrow$ similar to the $\mathbb{Z}$-graded Lie alg-s as discussed in Lectures 4–6

Shapovalov pairing $M_\lambda \times M_\lambda \longrightarrow \mathbb{C}$

coming from a more general map

$$\boxed{\mathcal{U}(n_-) \times \mathcal{U}(n_-) \longrightarrow \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]}$$

We shall define it now!

$$\underbrace{\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+}_{\Delta\text{-decomp}} \overset{\text{PBW}}{\Longrightarrow} \boxed{U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \left(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+\right)}$$

$\Big\downarrow$ projection on the first summand

$$\boxed{\pi : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})}$$

Note: $f_i \cdot e_j \in \mathfrak{n}_- U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n}_+$

$\sigma(f_i \cdot \sqcup) = \sqcup \cdot e_i$

$\sigma(h_i) = h_i$

Def: Consider an $U(\mathfrak{h})$-valued pairing

$$\langle\, ,\, \rangle : U(\mathfrak{g}) \times U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$$

$$\boxed{(x, y) \longmapsto \pi(\sigma(x) \cdot y)}$$

Rmk:
- $\langle \cdot , \cdot \rangle$ - symmetric (b/c π commutes with π)
- If $x \in U(\mathfrak{g})_{\mu_1}$, $y \in U(\mathfrak{g})_{\mu_2}$, $\mu_1 \neq \mu_2 \Rightarrow \langle x, y \rangle = 0$.
- If $y \in U(\mathfrak{g})\mathfrak{n}_+$ or $x \in U(\mathfrak{g})\mathfrak{n}_+ \Rightarrow \langle x, y \rangle = 0$

$\Rightarrow$

$\Rightarrow \langle \cdot , \cdot \rangle$ is determined by restriction $\boxed{U(\mathfrak{n}_-) \times U(\mathfrak{n}_-) \to U(\mathfrak{h})}$

As before, we shall treat restriction to each degree cpt

$$\boxed{\langle \cdot, \cdot \rangle^{\eta} : \mathcal{U}(\mathfrak{n}_-)_{-\eta} \times \mathcal{U}(\mathfrak{n}_-)_{-\eta} \longrightarrow \mathcal{U}(\mathfrak{h}).} \qquad \forall \eta \in Q^+$$

Rmk: For any $\lambda \in \mathfrak{h}^*$, evaluating $\mathcal{U}(\mathfrak{h}) \simeq \mathbb{C}[\mathfrak{h}^*]$ at λ provides a map

$$\mathcal{U}(\mathfrak{n}_-)_{-\eta} \times \mathcal{U}(\mathfrak{n}_-)_{-\eta} \longrightarrow \mathbb{C}$$

Def 2: Define the pairing $M_\lambda \times M_\lambda \longrightarrow \mathbb{C}$ via

$$\boxed{\langle u_1(\underbrace{v_\lambda}_{\text{h.wt. vector}}), u_2(v_\lambda) \rangle = \underbrace{\langle u_1, u_2 \rangle (\lambda)}_{\text{evaluation at } \lambda \text{ as in Rmk above.}}}$$

in $\mathcal{U}(\mathfrak{n}^-)$

Rmk: As before (for $\mathbb{Z}$-graded Lie alg-s)

$$\boxed{M_\lambda\text{-irred} \Longleftrightarrow \langle\cdot,\cdot\rangle_\lambda\text{ - nondeg.}}$$

Main Theorem: Let $\mathfrak{g}=\mathfrak{g}(A)$ be a Kac-Moody (or even more generally symm. contragredient alg.)

$$\det\left(\langle\cdot,\cdot\rangle^\eta\right) = \underbrace{\text{constant}}_{\neq 0} \cdot \prod_{\alpha>0} \prod_{n\geq 1} \left(h_\alpha + \rho(h_\alpha) - \frac{n(\alpha,\alpha)}{2}\right)^{P(\eta-n\alpha)}$$

($\alpha>0$ counted with multiplicities)

$P(\eta)$ = Kostant partition f-n = $\dim U(\mathfrak{n}_-)_{-\eta}$

Above: $\langle\cdot,\cdot\rangle : U(\mathfrak{n}_-)\times U(\mathfrak{n}_-) \longrightarrow U(\mathfrak{h})=S(\mathfrak{h})$

$\rho\in\mathfrak{h}^*$ - same as in Casimir

! **Important Conclusion**: $\boxed{M_\lambda\text{-irreducible} \Longleftrightarrow (\lambda+\rho)(h_\alpha)\neq \frac{n}{2}(\alpha,\alpha)\ \forall \alpha\in\Delta_+\ \forall n\geq 1}$

Proof of Theorem

Step 1: Computation of the leading order term.

Lemma 1: The leading term of $\det(\langle \circ, \circ \rangle^\eta) = \prod_{\alpha \in \Delta^+} \prod_{n \geq 1} h_\alpha^{P(\eta - n\alpha)}$.

roots are counted with multiplicities

Exercise

[Sketch of argument: basis of $\mathcal{U}(\mathfrak{n}_-)_{-\eta}$

$$\begin{array}{c} \mathcal{U}(\mathfrak{n}_-)_{-\eta} \\ \left(\begin{array}{ccc} \ddots & & \\ & \pi(\underbrace{\sigma(x_i)}_{\mathcal{U}(\mathfrak{n}_+)} \cdot \underset{\mathcal{U}(\mathfrak{n}_-)}{x_j}) & \\ & & \ddots \end{array} \right) \end{array}$$

$e_j\, e_i\, (f_k\, f_l)$

$f_k f_l e_i \xrightarrow{\pi} 0 \quad \ldots$

$\delta_{ik} h_i f_l + \delta_{il} f_k h_i \longmapsto 0.$]

Def: $\beta \in Q^+ \setminus \{0\}$ – quasiroot if $\exists \alpha \in \Delta^+$, s.t. β is proportional to α, to be denoted $\boxed{\beta \sim \alpha.}$

Step 2

Lemma 2: $\det(\langle \cdot, \cdot \rangle^{\lambda})$ is equal up to nonzero constant to a product of linear factors

$$\boxed{\{h_\beta + \rho(h_\beta) - \tfrac{1}{2}(\beta, \beta)\},\ \beta\text{-quasiroot}}$$

Note: This is already a good approximation, but we still will have to show that only $\beta \in \mathbb{Z}_+ \cdot \Delta^+$ are allowed, and also need to track multiplicities properly.

(Proof)

M_λ - is not irreducible $\iff$ M_λ has a singular vector (let's say of degree $\lambda-\beta$)

$$\Updownarrow$$

$$\boxed{M_{\lambda-\beta} \hookrightarrow M_\lambda} \qquad \beta \in Q^+ \setminus \{0\}$$

$$\Delta$$

Casimir Operator (• commutes with $\mathfrak{g}(A)$-actions • written via $\mathfrak{g}(A)$-action $\Downarrow$ intertwines)

Δ must act by the same constant!

(Key spot where the Casimir operator is used)

$$(\lambda, \lambda+2\rho) \overset{?}{=} (\lambda-\beta, \lambda-\beta+2\rho)$$

$$\Updownarrow$$

$$(\lambda+\rho, \beta) = \frac{1}{2}(\beta,\beta)$$

$$\Updownarrow$$

$$\boxed{(\lambda+\rho)(h_\beta) = \frac{1}{2}(\beta,\beta)}$$

So:

M_λ - not irred $\Longrightarrow \exists \beta \in Q^+_{\setminus\{0\}}$: $\boxed{(\lambda+\rho)(h_\beta) = \frac{1}{2}(\beta,\beta).}$ $(*)_\beta$

Therefore, if $\lambda \in \mathfrak{h}^*$ is such that $(\lambda+\rho)(h_\beta) \neq \frac{1}{2}(\beta,\beta)$

$\Downarrow$

$\Longleftarrow$ M_λ - irred.

$\mathbb{C}[\mathfrak{h}^*] = \mathcal{U}(\mathfrak{h}) \ni \det(\langle \cdot, \cdot \rangle^\eta)$ must have zeroes on the countable union of hyperplanes $\{(*)_\beta \mid \beta \in Q^+_{\setminus\{0\}}\}$

$\Downarrow$

$\det(\langle \cdot, \cdot \rangle^\eta)$ must factor as a product of some of those $(*)_\beta$. (β must be a quasiroot)

BUT Note: β must be a quasiroot, due to Lemma 1. ∎.

One more technical result.

$\boxed{\text{Lemma 3}}$ Let β be a quasiroot s.t. $\boxed{(\lambda+\rho)(h_\beta) = \frac{1}{2}(\beta,\beta)}$

& s.t. $\boxed{\begin{array}{ll} (\lambda+\rho)(h_\gamma) \neq \frac{1}{2}(\gamma,\gamma) & \forall \gamma \in Q^+ \setminus \{0,\beta\} \\ (\lambda-\beta+\rho)(h_\gamma) \neq \frac{1}{2}(\gamma,\gamma) & \forall \gamma \in Q^+ \setminus \{0\}. \end{array}}$

Then: $\boxed{\text{Ker}\,(M_\lambda \longrightarrow L_\lambda) \text{ is a direct sum of finitely many copies of } M_{\lambda-\beta}.}$

- $(\lambda-\beta+\rho)(h_\gamma) \neq \frac{1}{2}(\gamma,\gamma)$ $\forall \gamma \overset{\in Q^+}{\neq 0}$ $\Longrightarrow$ $M_{\lambda-\beta}$ -irred (see Lemma 2)

- $$\left.\begin{array}{l} M_\lambda^{\text{sing}}(\lambda-\beta) \subseteq M_\lambda(\lambda-\beta) \\ \quad \Psi \\ \forall v \rightsquigarrow M_{\lambda-\beta} \hookrightarrow M_\lambda \\ \qquad v^+_{\lambda-\beta} \longmapsto v \end{array}\right\} \Longrightarrow U = M_{\lambda-\beta}^{\oplus ?} \hookrightarrow M_\lambda, \quad ? = \dim M_\lambda^{\text{sing}}(\lambda-\beta)$$

$\rightsquigarrow$ consider the quotient $\underline{L = M_\lambda / U}$ $\longleftarrow$ $\boxed{\text{Claim}}$: it's irred, i.e. L_λ.

(Continuation of proof of Lemma 3)

Claim: L - irreducible

$\triangleright$ If not, then $L^{\mathrm{sing}} \neq \mathbb{C} v_\lambda \Rightarrow \exists \gamma \in Q^+ \setminus \{0\}$, $v \in L^{\mathrm{sing}}(\lambda-\gamma) \Rightarrow M_{\lambda-\gamma} \xrightarrow[\text{homom}]{\text{nonzero}} L$

But then comparing actions of Casimir operator (as in Lemma 2)

$$\Downarrow$$

$$(\lambda+\rho)(h_\gamma) = \frac{1}{2}(\gamma,\gamma)$$

$$\Downarrow$$

$$\gamma = \beta$$

But $L^{\mathrm{sing}}(\lambda-\beta)=0$ by the very construction of L $\Rightarrow$ $\lightning$

$\circ$

Claim $\Rightarrow$ Lemma 3 as we get $\mathrm{Ker}(M_\lambda \twoheadrightarrow L_\lambda) = U = M_{\lambda-\beta}^{\oplus ?}$

$\blacksquare$

$\boxed{\text{Step 3}}$: Jantzen's filtration

$\boxed{\text{Idea}}$: Keep track of multiplicity of zero by deforming our construction: $\mathbb{C} \rightsquigarrow \mathbb{C}[t]$

Define $\boxed{\tilde{\mathcal{U}}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}[t]}$ $\boxed{\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \otimes_{\mathbb{C}} \mathbb{C}[t]}$

Extending $\mathbb{C} \rightsquigarrow \mathbb{C}[t]$ everywhere

Fix $\nu \in \mathfrak{h}^*$ s.t. $\nu(h_\alpha) \neq 0$ $\forall \alpha \in Q^+ \setminus \{0\}$.

$$\Downarrow$$

$$\tilde{\mathfrak{h}}^* \ni \boxed{\tilde{\lambda} := \lambda + t \cdot \nu}$$

Apply all our previous consideration in this extended setup.

Extend $\sigma \circlearrowright \mathcal{U}(\mathfrak{g})$ to $\sigma \circlearrowright \tilde{\mathcal{U}}(\mathfrak{g})$ via $\underset{x}{\overset{\Psi}{}} \otimes F(t) \mapsto \sigma(x) \otimes F(t)$

So: Have extended versions of the pairings $\widetilde{\langle \cdot, \cdot \rangle}$ and $\widetilde{\langle \cdot, \cdot \rangle}^{?}_{\tilde{\lambda}}$

$$M_\lambda \rightsquigarrow \tilde{M}_{\tilde{\lambda}} \text{ - Verma } / \tilde{\mathcal{U}}(\mathfrak{g}).$$

$M_\lambda \circlearrowleft \mathcal{U}(\mathfrak{g})$

Have: $\tilde{\mathcal{U}}(\mathfrak{g}) \xrightarrow{t \mapsto 0} \mathcal{U}(\mathfrak{g}) \rightsquigarrow \boxed{\tilde{\mathcal{U}}(\mathfrak{g}) \curvearrowright M_\lambda}$

$$\Rightarrow \boxed{\tilde{M}_{\tilde{\lambda}} \overset{p}{\twoheadrightarrow} M_\lambda \text{ - homom. of } \tilde{\mathcal{U}}(\mathfrak{g})\text{-modules}}$$

Key construction for today:

Consider the $\tilde{\mathcal{U}}(\mathfrak{g})$-module filtration of $\tilde{M}_{\tilde{\lambda}}$:

$$\tilde{M}_{\tilde{\lambda}} = \tilde{M}^{(0)} \supseteq \tilde{M}^{(1)} \supseteq \tilde{M}^{(2)} \supseteq \dots$$

$$\boxed{\tilde{M}^{(k)} = \{ v \in \tilde{M}_{\tilde{\lambda}} \mid \widetilde{\langle v, w \rangle} \vdots t^k \ \forall w \in \tilde{M}_{\tilde{\lambda}} \}.}$$

Def: Jantzen $\mathcal{U}(\mathfrak{g})$-module filtration of M_λ is obtained by applying $p: \tilde{M}_{\tilde{\lambda}} \twoheadrightarrow M_\lambda$ to $\tilde{M}^{(k)}$ $\rightsquigarrow \boxed{M_\lambda = M_\lambda^{(0)} \supseteq M_\lambda^{(1)} \supseteq M_\lambda^{(2)} \supseteq \dots}$

$\boxed{\text{Recall:}}$ $M^{(1)}_\lambda = \mathrm{Ker}(\langle \circ , \circ \rangle_\lambda)$.

Step 4

Know: $\det(\langle \circ , \circ \rangle^\eta)$ is a product $\left\{ h_\beta + \rho(h_\beta) - \frac{(\beta,\beta)}{2} \;\middle|\; \beta \sim \alpha \in \Delta^+ \right\}$.
proportional

Fix $\alpha \in \Delta^+$

- Case 1: $(\alpha,\alpha) = 0 \Rightarrow (\beta,\beta) = 0 \Rightarrow h_\beta + \rho(h_\beta) - \frac{(\beta,\beta)}{2} \sim h_\alpha + \rho(h_\alpha) - \frac{(\alpha,\alpha)}{2}$

$\Downarrow$

the leading term from Lemma 1 guarantees the total multiplicity of above factor $= \sum_{n \geq 1} P(\eta - n\alpha)$.

So: For $\alpha \in \Delta^+$ s.t. $(\alpha,\alpha) = 0$, the power of $h_\alpha + \rho(h_\alpha)$ in $\det(\langle \cdot , \cdot \rangle^\eta)$ is as claimed in Theorem.

- $\underline{\text{Case 2}}$: $(\alpha,\alpha)\neq 0$.

Note that as $\beta\in Q^+ \Rightarrow \beta = r\alpha$, $r\in\mathbb{Q}$.

$\boxed{\text{Exercise}}$ If $(\alpha,\alpha)\neq 0$, $\beta\in\mathbb{Q}_+\cdot\alpha \Rightarrow \exists\,\lambda\in\mathfrak{h}^*$ s.t.

$$(\lambda+\rho)(h_\beta)=\tfrac{1}{2}(\beta,\beta),\quad \underset{\forall\gamma\neq 0,\beta}{(\lambda+\rho)(h_\gamma)\neq\tfrac{1}{2}(\gamma,\gamma)},\quad \underset{\forall\gamma\neq 0}{(\lambda-\beta+\rho)(h_\gamma)\neq\tfrac{1}{2}(\gamma,\gamma)}$$

- Pick λ as in the Exercise

$\rightsquigarrow$ consider Jantzen's filtration on the corresponding Verma

$$\boxed{M_\lambda = M^{(0)} \not\supseteq M^{(1)} \supseteq M^{(2)} \supseteq \dots}$$

- $\underline{\text{Lemma 3}} \Rightarrow M^{(1)} \simeq \underbrace{M_{\lambda-\beta}^{\oplus ?}}_{\substack{\text{must be irred.}\\ \text{as } (\lambda-\beta+\rho)(h_\gamma)\neq\frac{1}{2}(\gamma,\gamma)\ \forall\gamma\in Q^+\setminus\{0\}}} \overset{\forall i\geq 1}{\Longrightarrow} M^{(i)} \simeq M_{\lambda-\beta}^{\oplus N_i}$ (b/c $M_{\lambda-\beta}$ is irreducible).

deeper filtration pieces

- Set $\boxed{N_\beta = \sum_i N_i}$

But then (by the very definition of Jantzen's filtration):

$$\boxed{\det(\widetilde{\langle \circ, \circ \rangle}^{\eta}_{\bar{\lambda}}) \text{ is divisible by exactly } \sum_i \dim M^i(\lambda-\eta) = N_\beta \cdot P(\eta-\beta)\text{-th power of } t}$$

$$\Downarrow$$

$$\boxed{\det(\langle \circ, \circ \rangle^{\eta}) \text{ is divisible exactly by } N_\beta \cdot P(\eta-\beta)\text{-th power of } h_\beta + \rho(h_\beta) - \tfrac{1}{2}(\beta,\beta)}$$

Thus: the multiplicity of h_α in the leading term of $\det(\langle \circ, \circ \rangle^{\eta})$ equals:

$$\sum_{\beta \sim \alpha} N_\beta \cdot P(\eta-\beta)$$

But: the same multiplicity $= \sum_{n\geq 1} P(\eta - n\cdot\alpha)$ by Lemma 1

$$\Rightarrow \boxed{\sum_{\beta\sim\alpha} N_\beta \cdot P(\eta-\beta) = \sum_{n\geq 1} P(\eta - n\alpha)}$$

Exercise: Functions $\{\phi_\beta(\eta) = P(\eta-\beta)\}_{\beta\in(\mathbb{Q}_+\cdot\alpha)\cap(Q^+\setminus\{0\})}$ are lin. ind.

Hence: • $N_\beta = 0$ if $\beta \notin \mathbb{Z}_{>0}\cdot\alpha$

• $N_{n\alpha} = \dim \mathfrak{g}_\alpha$

This finally completes the proof of Theorem. ✓