Lecture #3 09/02/2016

- Organizational:
  - Please always write your recitation number on all hwks & exams
  - If there are someone who has not yet received an ordered textbook, you can take a photo of problems / printed handouts after this lecture.
  - In your hwk the first problem is just on understanding how we draw schedules, the last problem is on verification of some criteria for some voting methods, while the other 4 problems ask you to find a complete ranking in 1 election using 4 different voting methods (we already covered 3 and we will learn 1 today)

- Last time: "Borda count" and "Plurality-with-Elimination"

Roughly speaking, the key features of these methods are:

- Borda count attributes pts to all candidates (from 1pt to N pts) (N = number of candidates)
- In the Plurality-with-Elimination, we step-by-step use the classical Plurality Method to find the candidat with the fewest votes; we cross that candidate out and repeat procedure

Rmks:

1. If we find a candidate with >50% votes even before the only 2 are left, then we can immediately stop the plurality-with-elimination and announce this candidate to be the winner
2. In the plurality-with-elimination we MUST use the preference ballots, but each round we count only 1st preference (out of the remaining, not yet crossed-out)
3. After we crossed out, we can still determine who is the most preferable out of the remaining on each ballot.

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• Examples for the methods from last lecture.

Consider the same example of high-school election as we had in Lecture 1

* Borda count

Number of Voters	15	10	11	4
1^{st}	(60) A	(40) C	(44) D	(16) B
2^{nd}	(45) B	(30) B	(33) B	(12) C
3^{d}	(30) C	(20) D	(22) A	(8) D
4^{th}	(15) D	(10) A	(11) C	(4) A

Summing up we see that:

A gets $60+10+22+4=96$ points

B gets $45+30+33+16=124$ points

C gets $30+40+11+12=93$ points

D gets $15+20+44+8=87$ points

Outcome: B is the winner, while the complete ranking is #1: B, #2: A, #3: C, #4: D.

Note: If using the Plurality method the ranking is #1: A, #2: D, #3: C, #4: B

* Plurality - with - elimination

Let us now apply the plurality-with-elimination method (we already did this fast last time)

! Discuss in details !

Number of Voters	15	10	11	4
1^{st}	A	C	D	B
2^{nd}	B	B	B	C
3^{d}	C	D	A	D
4^{th}	D	A	C	A

Round 1

A	B	C	D
15	4	10	11

So: B has the fewest votes
$\Rightarrow$ eliminate B and cross him out in schedule

Round 2

A	C	D
15	14	11

Here C is counted 14 times in total, since it is #1 choice on 10 ballots and it is #2 choice on 4 ballots who chose B as their 1^{st} preference and who is crossed out B after Round 1

eliminate D and cross out D $\Leftarrow$

Round 3

A	C
26	14

Here A is counted 26 times since it appears as #1 choice on 15 ballots and as the next-after (crossed out) B & D on 11 ballots $(15+11=26)$

So we see that applying the plurality-with-elimination method to our example A is the winner, while the complete ranking is #1: A, #2: C, #3: D, #4: B.

- The method of Pairwise Comparisons

The last method we will learn is the "Pairwise Comparisons's method"

Idea: As the name suggests, we consider all pairs (out of N candidates) X & Y and compute how many voters prefer X to Y. (This method essentially uses the preference ballots!) If X is ranked above Y, then we give 1 point to X and 0 to Y; if Y is ranked above X, then we give zero (0) points to X and one (1) point to Y; if they are equally preferred we'll give $\frac{1}{2}$ points to each (in hwk & exams there will be no such cases).

- Step 1: Compare each pair X vs Y and give (1,0), (0,1), or $(\frac{1}{2}, \frac{1}{2})$ points.
  ↳ Unless ties occur, find the winner and write it to the right column
- Step 2: Look at the right column showing winners in each pair and find which candidate appears most frequently

Outcome: Declare this candidate to be the winner!

Rmk:
- If there are only 2 candidates, then we have just 1 pair.
- If there are 3 candidates (call them A, B, C), then we have 3 different pairs: A&B, A&C, B&C.
- If there are 4 candidates (call them A, B, C, D), then we have 6 different pairs: A&B, A&C, A&D, B&C, B&D, C&D.
- In general, for N candidates there are $\frac{N \cdot (N-1)}{2}$ different pairs.

* Let us illustrate how this method works on our example from 1st Lecture

Number of Voters	15	10	11	4
1st	A	C	D	B
2nd	B	B	B	C
3d	C	D	A	D
4th	D	A	C	A

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Pairwise Comparison	Votes	Winner
A vs B	A(15), B(25)	B
A vs C	A(26), C(14)	A
A vs D	A(15), D(25)	D
B vs C	B(30), C(10)	B
B vs D	B(29), D(11)	B
C vs D	C(29), D(11)	C

P.S. Discuss this in more details in class!

Hence, we see that B is the winner (as it appears more times than any other candidate), while there is a tie b/w A, C, D. As we never discussed how to resolve ties, let us just grant the 2nd place to all three of them.

Upshot:
- While this method utilizes the key feature of the preference ballot "the ability to compare any 2 candidates on everyone's ballot" it is very time consuming. However, it is "more fare" than "Plurality"
- It is very similar to the round-robin tournament, where there is a game between any 2 teams and in each game we give 1 point to the winner and 0 points to the other team (in the rare case of ties, we give $\frac{1}{2}$ points to each team). The winner is the team with the most points.
- In this method, the winner is always preferred by a majority of voters (see the end of § 1.2 in textbook, where such candidates are called Condorcet candidate)

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- So far we discussed 4 different methods (there are many more which we do not have time to discuss).

Question: Which is the best method out of all those?

Though it seems to be a very simple question to ask, it took centuries to understand that there is no apriori best method. This understanding goes back to 1940s to the work of US economist Kenneth Arrow.

Arrow asked himself: "What is required of the voting method so that it is at least fair?"

This led him to set a minimum set of requirements, called Arrow's fairness criteria. The simplified list of those is as follows:

- The majority criteria: The candidate with a majority of the 1st place votes should always be the winner.
- The monotonicity criterion: If candidate X is the winner, then X would still be the winner had a voter ranked X higher in his preference ballot.
- The independence-of-irrelevant-alternatives (IIA) criterion: If candidate X is the winner, then X would still be the winner had one or more of the losing candidates not been in the race.

These three fairness criteria represent some of the basic principles a democratic election should have. If a method violates any one of these criteria, then there is a potential for unfair outcome.

* Let us now see how some of the methods we know violate some of the above criteria.

Example 1: The Borda count method violates the majority criterion (see example on top of p. 2).

Example 2: The plurality-with-elimination method violates the monotonicity criterion.

Following example 1.21 from the textbook, consider the schedule

Number of voters	7	8	10	2
1^{st}	A	B	C	A
2^{nd}	B	C	A	C
3^{d}	C	A	B	B

Round 1

A	B	C
9	8	10

$\Rightarrow$ eliminate B $\Rightarrow$ 8 ballots go to C

Round 2

A	C
9	18

$\Rightarrow$ the winner is C.

Let us now change the schedule by moving C above A in 4^{th} column

	7	8	10	2
1^{st}	A	B	C	C
2^{nd}	B	C	A	A
3^{d}	C	A	B	B

Round 1

A	B	C
7	8	12

$\Rightarrow$ eliminate A $\Rightarrow$ 7 ballots go to B

Round 2

B	C
15	12

$\Rightarrow$ the winner is B

So: the monotonicity criterion fails.

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This week we discussed elections and mathematics behind all of them.

- We started by specifying the key elements of all elections: candidates, voters, ballots

candidates, voters, ballots $\Downarrow \leftarrow$ voting method

outcome

- We discussed the four most common voting methods:
  - Plurality
  - Borda count
  - Plurality-with-elimination
  - Pairwise comparison

And we witnessed that applying different methods to the same elections' data, we obtain completely different outcome.

- We discussed some of the criteria each fair method should satisfy

It turns out, it is mathematically impossible for any voting method to satisfy all the fairness criteria (under assumption that there are more than 2 candidates). This result is known under the name "Arrow's Impossibility Theorem".