Lecture #4

09/07/2016

- Before we switch to a new topic, let me briefly summarize the material from the end of previous lecture, since we rushed through by the very end.

- In 1940's Kenneth Arrow proved the "Arrow's Impossibility Theorem", which says that it is mathematically impossible for any voting method to satisfy all the fairness criteria under an assumption that #candidates > 2

  The three most basic fairness criteria were mentioned last time:

  * The majority criterion: The candidate with a majority of the 1st place votes should always be the winner.

  * The monotonicity criterion: If candidate X is the winner, then X would still be the winner had any voter ranked X higher in his preference ballot.

  * The independence-of-irrelevant-alternatives (IIA) criterion: If candidate X is the winner, then X would still be the winner had one or more of the losing candidates not been in the race

  The two examples from Friday's notes illustrate:

  - the Borda count method violates the majority criterion
  - the plurality-with-elimination method violates the monotonicity criterion.

- Ask if anyone has any questions on the material covered last week on the topic "Mathematics of Elections".

  Do not be shy: come to the office hours, which are designated for you to resolve any questions on the material.

- Organizational:
  - You should leave your hwk on my desk by the end of this lecture.
  - Please write down your recitation number on hwk and put in the corresponding pile.
  - No late hwks submitted after 10a.m. of today will be accepted.
  - I apologize for the fact that one of the hwk problems involved ties. Following my announcement on Blackboard you should list all candidates who share the 1st, 2nd, etc. places.

- Questionnairre

Please fill out the questionnaires and submit them at the end of today's lecture. They are anonymous and optional, but I would really appreciate if you could indicate any suggestions on my lectures, which I will try to incorporate in the rest of the class.

- New topic "The Mathematics of Power"

For the rest of this week we will be covering the material of Section 2 from Textbook.

The new homework will be posted today and will be due by the end of the next Wednesday's lecture.

Lecture #4 09/07/2016

"The Mathematics of Power"

Underlying Idea

In the previous discussion of Elections, we always assumed that the votes from different voters have equal impact on the outcome. This principle is sometimes called "one voter - one vote" and it holds only in the democracy.

However, this is not always the case as different voters can have different influence on the outcome. This principle can be described as "one voter - x votes" and it is called a "weighted voting"

Examples:
- shareholder votes in corporations
- business partnerships
- Electoral College (electors chosen by states, who vote for the election of the president, vice president)
- United Nations Security Council

- In the first example, each of the shareholders can participate in all elections concerning management of the corporation. However, it is clear that the vote of the shareholder holding 100 shares should be more influential than a vote from the shareholder holding 1 share.
- In the second example, we may have a business organized by 4 friends which invested different amounts of $ to start up this business, say 10K, 8K, 6K, 4K. Then it is clear that the person who invested 10K should be more influential than a person who invested 4K.
- In the third example, each state of the US has several representatives which represent the interests of the state and vote in the same way. Therefore, we can think of the Electoral College as having 51 (= # states) voters, each with a weight determined by the size of the delegation, e.g. California has 55 electoral votes, while Montana has only 3.

Clearly the vote of California is much stronger than the one by Montana

- We will consider the 4^{th} example later on.

Lecture #4

09/07/2016

- Basic elements of the weighted voting system with only 2 choices Yes/No: (!)

  - **Players** (this stays for voters, e.g. individuals, institutions, corporations, states, countries, etc.)

    We will use N to denote the number of players and we will call the players: $P_1, P_2, \ldots, P_N$ (P_k stays for the "player k")

  - **The weights** ← this is the key information

    The weight of the player is the number of votes controlled by this player.
    We denote the weights of $P_1, \ldots, P_N$ by $w_1, \ldots, w_N$ (assuming $w_1, \ldots, w_N \geq 0$).
    We will use V to denote the total number of votes in the system, i.e.

    $\boxed{V = w_1 + w_2 + \ldots + w_N}$.

  - **The quota**

    This is the minimum number of votes required to pass a motion.
    We denote the quota by q. For example, the quota can be a simple majority of the votes, or 60% of votes, or 75% of votes, or all votes.

Notation: The above data is usually summarized by the following notation:

$$\boxed{[q : w_1, w_2, \ldots, w_N]} \quad (\text{where } w_1 \geq w_2 \geq \ldots \geq w_N)$$

we reorder the players accordingly

Key Question we will discuss: How can we measure a player's power in the weighted voting system?

Lecture #4 09/07/2016

- Basic examples (with associated terminology)

① Consider example 2 from p.2. We have a new business organized by 4 friends who invested 10K, 8K, 6K, 4K. Each of them got 1 share for each \$K spent $\Rightarrow$ 1^{st} got 10 shares, 2^{nd} got 8 shares, 3^{d} – 6 shares, 4^{th} – 4 shares.
Assume that the quota is two-thirds of the total number of votes.
Since $V = 10+8+6+4 = 28$ and $\frac{2}{3}V = \frac{2}{3}\cdot 28 = 18\frac{2}{3}$, the actual quota is $q=19$.

Using the weighted voting system notation, this partnership can be described mathematically as

$$[19: 10, 8, 6, 4].$$

② Consider the above setting from ①, but set $q=14$.
Then if P_1 and P_4 vote "Yes", while P_2 and P_3 vote "No", then both "Yes" and "No" win, since $w_1 + w_4 = w_2 + w_3 = 14 = q$.
This is a mathematical model of "anarchy".
In what follows, we will assume $\boxed{q > \frac{1}{2}\cdot V.}$

③ Consider the setting from ①, but set $q=29$.
Since $V = w_1 + w_2 + w_3 + w_4 = 28 < q$, no motion would pass and nothing will be done.
This is a mathematical model of "gridlock"
In what follows, we shall assume $\boxed{q \le V}$

④ Let us consider the setting from ①, but set the quota $q=25$.
The corresponding weighted voting system is $[25: 10, 8, 6, 4]$.
Note that the motion will be passed only if all players vote for it. In particular, if P_1, P_2, P_3 vote for it, but P_4 does not, then there will not be enough votes to get the quota.
So: Even though w_4 is much smaller than w_1, but as we see P_4 has the same power as P_1 (and as P_2, P_3)

! Example ④ shows that a player with small number of votes can be very powerful.

⑤ Let us now consider a modification of the example ①, where $w_1 = w_2 = w_3 = 100$, $w_4 = 99$ (i.e. P_1, P_2, P_3 invested 100K, P_4 – 99K) and $q = 300$.
The corresponding weighted voting system can be described as

$$[300 : 100, 100, 100, 99].$$

Note that for a motion to pass P_1, P_2 and P_3 have to vote "Yes", while the vote of P_4 does not matter at all.

! Example ⑤ shows that a player with large number of votes can have no powe

⑥ Let us now modify the numbers again, so that the wighted voting system is $[90 : 100, 10, 10, 10]$.
Note that if P_1 votes for the motion, then it will pass, while if P_1 is against it, the motion will fail
This is a mathematical model of "dictatorship".

! If $w_1 \geq q > w_2 + w_3 + \ldots + w_N$, then P_1 is called a "dictator"

⑦ Consider the weighted voting system from ① : $[19 : 10, 8, 6, 4]$.
Note that $8+6+4 = 18 < q$, so the motion will not pass without the support of P_1. Note also that $w_1 = 10 < q \Rightarrow P_1$ is not a dictator.
In this case, we call P_1 the "spoiler" and will say that P_1 has a veto power.

! A player P with weight w has a veto power if $w < q$, $V - w < q$.

Upshot: We considered 7 completely different and instructive examples which show that the power is not actually proportional to the number of votes, i.e. weights can be really deceiving.