Last time we started a new topic "Mathematics of Power"
We will still have elections (with only 2 candidates / 2 options / "Yes" vs "No"), but these elections will no longer be democratic.

- **Def**: We will call such elections / voting as "weighted voting" (i.e. different voters have different weights of their votes).
- The key elements of any "weighted voting" are:
  - Players (we usually denote them by $P_1, \ldots, P_N$, where $N = \#$ players)
  - Weights (each player has a weight which is equal to the number of votes controlled by this player)

    The weights of $P_1, \ldots, P_N$ will be denoted by $w_1, \ldots, w_N$

    We use V to denote the total number of votes, i.e. $\boxed{V = w_1 + \ldots + w_N}$
  - Quota q (this is the minimum number of votes required to pass a motion)
- Notation: The above data is usually summarized as:

$$[q : w_1, w_2, \ldots, w_N] \quad (\text{where } w_1 \geq w_2 \geq \ldots \geq w_N)$$

N.B. We reorder the players accordingly so that their weights are non-increasing

Basic Question: How can we measure a player's power in the weighted voting system?

Naive Suggestion you might have is that the power is proportional to the weights of players.

! However, that's not true. As we will see in a moment there are cases when players with small weight have a lot of power as well as the cases when players with large weights have almost no power

- Basic examples (with associated terminology)

① (Consider example 2 from p.2) We have a new business organized by 4 friends who invested 10K, 8K, 6K, 4K. Each of them got 1 share for each $K spent ⇒ 1^{st} got 10 shares, 2^{nd} got 8 shares, 3^{d} – 6 shares, 4^{th} – 4 shares.
Assume that the quota is two-thirds of the total number of votes.
Since $V = 10+8+6+4 = 28$ and $\frac{2}{3}V = \frac{2}{3}\cdot 28 = 18\frac{2}{3}$, the actual quota is $q=19$.
Using the weighted voting system notation, this partnership can be described mathematically as

$$[19 : 10, 8, 6, 4].$$

② Consider the above setting from ①, but set $q=14$.
Then if P_1 and P_4 vote "Yes", while P_2 and P_3 vote "No", then both "Yes" and "No" win, since $w_1 + w_4 = w_2 + w_3 = 14 = q$.
This is a mathematical model of "anarchy".
In what follows, we will assume $\boxed{q > \frac{1}{2}\cdot V.}$

③ Consider the setting from ①, but set $q=29$.
Since $V = w_1 + w_2 + w_3 + w_4 = 28 < q$, no motion would pass and nothing will be done.
This is a mathematical model of "gridlock"
In what follows, we shall assume $\boxed{q \le V}$

④ Let us consider the setting from ①, but set the quota $q=25$.
The corresponding weighted voting system is $[25 : 10, 8, 6, 4]$.
Note that the motion will be passed only if all players vote for it. In particular, if P_1, P_2, P_3 vote for it, but P_4 does not, then there will not be enough votes to get the quota.
So: Even though w_4 is much smaller than w_1, but as we see P_4 has the same power as P_1 (and as P_2, P_3)

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! Example ④ shows that a player with small number of votes can be very powerful.

⑤ Let us now consider a modification of the example ①, where $w_1 = w_2 = w_3 = 100$, $w_4 = 99$ (i.e. P_1, P_2, P_3 invested 100K, P_4 - 99K) and $q = 300$.
The corresponding weighted voting system can be described as

$$[300 : 100, 100, 100, 99].$$

Note that for a motion to pass P_1, P_2 and P_3 have to vote "Yes", while the vote of P_4 does not matter at all.

! Example ⑤ shows that a player with large number of votes can have no power

⑥ Let us now modify the numbers again, so that the wighted voting system is $[90 : 100, 10, 10, 10]$.
Note that if P_1 votes for the motion, then it will pass, while if P_1 is against it, the motion will fail

This is a mathematical model of "dictatorship".

! If $\boxed{w_1 \geq q > w_2 + w_3 + \dots + w_N}$, then P_1 is called a "dictator"

⑦ Consider the weighted voting system from ① : $[19 : 10, 8, 6, 4]$.
Note that $8+6+4 = 18 < q$, so the motion will not pass without the support of P_1. Note also that $w_1 = 10 < q \Rightarrow P_1$ is not a dictator.

In this case, we call P_1 the "spoiler" and will say that P_1 has a veto power.

! A player P with weight w has a veto power iff $\boxed{w < q,\ V - w < q.}$

Upshot: We considered 7 completely different and instructive examples which show that the power is not actually proportional to the number of votes, i.e. weights can be really deceiving!

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Before we switch to different ways of measuring power, let us consider couple of simple exercises which are similar to the first two from hwk.

Problem 1: Consider the weighted voting system $[q: 10, 8, 5]$.

(a) Find the smallest value of q for which all three players have veto power

(b) Find the smallest value of q for which P_1 has a veto power, but P_3 does not have a veto power

(a) We need to have

- $10<q$ & $8+5=13<q$ to guarantee that P_1 has a veto power.
- $8<q$ & $10+5=15<q$ to guarantee that P_2 has a veto power
- $5<q$ & $10+8=18<q$ to guarantee that P_3 has a veto power.

Hence, the answer to part (a) should be the smallest integer which is larger than $10, 13, 8, 15, 5, 18 \Rightarrow q=19$ is the smallest quota.

(b) As in (a), P_1 has a veto power iff $q>10$ & $q>13 \Leftrightarrow q \geq 14$.
Analogously P_3 has a veto power iff $q>5$ & $q>18 \Leftrightarrow q \geq 19$.

Hence, the answer is $q=14$.

Problem 2: Consider the weighted voting system $[q: 10, 6, 5, 4, 2]$.

(a) What is the smallest value that the quota q can take?

(b) What is the largest value that the quota q can take?

(c) What is the value of the quota if more than two-thirds of the votes are required to pass a motion?

Recall $\frac{V}{2} < q \leq V$. In our case $V = 10+6+5+4+2 = 27$.

(a) $q > \frac{27}{2} = 13\frac{1}{2} \Rightarrow q \geq 14$. Answer: $q=14$

(b) $q \leq V = 27$. Answer: $q=27$.

(c) $q > \frac{2}{3}V = \frac{2}{3} \cdot 27 = 18 \Rightarrow q \geq 19$. Answer: $q=19$