Lecture #7 09/14/2016

- - The new Hwk will be posted today and is due next Wedn.
  - You shall go to your recitations to pick up graded hwk
    ↳ If you think something is graded in an unfair way, please appeal!
  - The dates for the review sessions have been posted
    ↳ last chance for you to ask q-s before exams.

- Last time we learned the notion of the Banzhaf power index, which is one of the possible ways to compare the power.

Rmks:
- The coalition consisting of all players is usually called "the grand coalition".
- Under our assumptions ($q \le V$ and there are no dictators), the grand coalition is always winning, while all coalitions consisting of 1 player are losing.
- It is an instructive simple exercise (left to the audience) to understand why $T \neq 0$, i.e., at least one of the players has a non-zero critical count.
  (otherwise we might face a problem of dividing by T)
- The total number of coalitions on N players is $2^N - 1$ (see last page of previous lecture's notes).

This completes our discussion of the "Mathematics of Power".
If anyone has q-s, please come to OH.

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- Today: "The Mathematics of Sharing" [Section 3 of Textbook]
(next homework will be entirely on that subject).

The problems we will discuss are usually known under the name
fair-division problems.

We will start our discussion by introducing the basic terminology and concepts of fair division. Once we have all the terminology, we will discuss the most common approaches and illustrate them with examples
(As in previous lectures, we cover only the top of the iceberg)

- Basic Elements of a Fair-Division Game

– Assets (This includes all valuables/goodies we want to divide, e.g. candies, real estate, shares, jewelry, etc. Note: The assets can also be of "negative value", e.g. obligations, liabilities)
denoted by S

– Players (The parties parties which are dividing the assets from above. In our examples, these will be individuals most of the time, but in general these might be some groups, institutes, or even countries)
denoted $P_1, \dots, P_N$

– Value system (This is the key information! Each player has his own value system, i.e. the ability to give a value to any part of the assets)

Once we have those 3 basic elements of the Fair-Division Game, we want to divide the assets among the players, taking into account their value systems, so that "everyone is happy".
This requires the last element:

– Fair-division Method (These are the rules we are gonna use to divide the assets among the players)
there are many different methods, in the same way as we had many different voting methods.

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! We will also make the following assumptions:

- Rationality (each player tries to maximize his/her share of the assets)
- Cooperation (the players accept the rules of the game; the rules are such that after a finite number of moves the game terminates with the division of assets)
- Privacy (Players do not have any useful information on the other player's value systems. Note: In the real life that does not always hold)
- Symmetry (Players have equal rights in sharing the assets)

- Fair Shares and Fair Divisions

We already spoke about "fair division" of the asset S among players $P_1, \ldots, P_N$.

Q: What does it actually mean?

To answer this natural q-n, we need to introduce new concepts:

- Fair share (w.r.t. a player P)

  A share s of the assets S is called a proportional fair share to P if in P's opinion, the value of s is at least $\frac{1}{N} \cdot$(Value of S), where $N = \#$players

- Fair division (of the set of assets S among N players $P_1, \ldots, P_N$)

  If we managed to divide the set of all assets S into N shares $s_1, \ldots, s_N$ and assign s_1 to P_1, s_2 to P_2, $\ldots$ s_N to P_N so that each player got a fair share, then we say that we have achieved a fair division of the assets S between $P_1, \ldots, P_N$

Def: A fair-division method is a set of rules that guarantees that at the end of the game each player will have received a fair share

- Before we proceed to the fair-division methods, let us provide a simple example to illustrate the notion of fair share & fair division.

Example 1: Andrew, Bob, and Carol are dividing among themselves a set of common assets equally owned by the three of them. The assets are divided into three shares s_1, s_2, s_3 with the values of the shares to each player expressed as a percent of total value of the assets:

	s_1	s_2	s_3
Andrew	38%	28%	34%
Bob	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Carol	34%	40%	26%

Note: The sum in each row should equal 100%, but the sum in columns is arbitrary.

Q-s: (a) Which of the shares are fair for Andrew?
(b) —//— Bob?
(c) —//— Carol?
(d) Find all possible fair divisions of the assets using s_1, s_2, s_3 as shares.
(e) Of the fair divisions from (d), which one is the best?

(a) As $38 > \frac{100}{3}$, $34 > \frac{100}{3}$, $28 < \frac{100}{3}$, we see that s_1, s_3 are fair for Andrew.
(b) All shares are fair for Bob, since each of them is exactly $\frac{100}{3}$.
(c) As $34 > \frac{100}{3}$, $40 > \frac{100}{3}$, $26 < \frac{100}{3}$, we see that s_1, s_2 are fair for Carol.
(d) Andrew should get s_1 or s_3. If Andrew gets s_1, then Carol must get s_2, and then Bob gets s_3. This is a fair division.

If Andrew gets s_3, then Carol can either get s_1 or s_2 and then Bob gets s_2 or s_1, respectively. Both options – fair division.

(1) A – s_1, B – s_3, C – s_2; (2) A – s_3, B – s_2, C – s_1; (3) A – s_3, B – s_1, C – s_2

(e) It is clear that the best of those three is (1), since Bob always gets $33\frac{1}{3}\%$, while Andrew & Carol get the most valuable shares w.r.t. their own value systems ∎

Rmk: It is not always possible to find the best (or even any) fair division.