- All homeworks will be handed out at your recitations. If you miss one, you should be able to pick them up on the next, or make special arrangements with your recitation leaders (I do not have any of the graded hwks)

- You are highly encouraged to appeal in case you think you provided a correct proof, but got a low grade

! In your homework assignments and tests, you are supposed to provide an argument, not only the final answer. In case you present a correct answer, but without any indicated computations we will not grant the full grade!

Last time we started a new top "Mathematics of Sharing".
Each problem we face will contain 3+1 basic elements:

- Assets
- Players
- Value system

} Basics, part of formulation

and

- Fair-division Method ← Today & on Monday we will learn some of the most common methods

Last time I forgot to mention the key assumptions (see p.3 of Lecture 7):
- Rationality
- Cooperation
- Privacy
- Symmetry

Lecture #8 09/16/2016

• Now that we have introduced all the basic elements of a fair-division game and discussed the notion of fair shares and fair divisions, it is TIME to learn some fair-division methods!

But before we start, let me point out a crucial difference between two different types of assets:

– Continuous

This means the set of assets S is divisible in infinitely many ways.

Typical examples: land, cake, pizza, ...

– Discrete

This refers to the case when the set of assets S is formed by indivisible objects.

Typical examples: jewelry, cars, houses, candies, ...

• The "Divider – Chooser Method"

We start from the simplest example: 2 players & continuous assets.

In this case, the most commonly used method is the following.

Out of the two players we have to assign who is the "Divider" and who is the "Chooser". (Use a coin to make this choice!)

Step 1: The divider devides S into two shares (which are of equal value w.r.t. his value system)

Step 2: The chooser picks the share out of those two which is a fair share in his value system.

Step 3: The remaining share goes to the divider.

! This is a fair-division method since the divider gets a share, whose value equals exactly $\frac{1}{2}\cdot$(Value of all assets S), while the chooser gets a share whose value is at least $\frac{1}{2}\cdot$(Value of all assets)

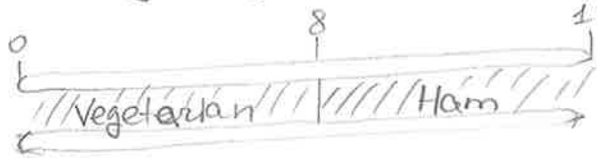
! Rmk: So it is always better to be a chooser.

Lecture #8

08/16/2016

• [illegible] Let us illustrate how the "Divider-Chooser Method" works and why it is always better to be a chooser.

Example 1: Two high-school students, Alice and Bob, jointly bought a half ham-half vegeterian 16 inches-long sub



Assume that Bob is a strict vegeterian and does not eat ham at all while Alice prefers the ham part twice more than the vegetarian part

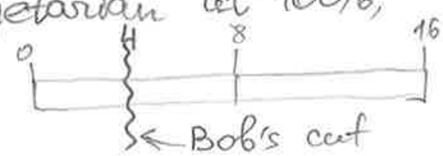
(a) Apply the "divider-chooser" method with Bob being a divider. Specify the cut and the value of the share which goes to Alice

(b) Apply the "divider-chooser" method with Alice being a divider. Specify the cut and compute the value of share that goes to Bob

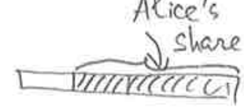
Assume: the cut is made perpendicular to the length.

► (a) Bob evaluates Vegetarian at 100%, ham at 0%.

Hence he divides



It is clear that Alice will take the rightmost part

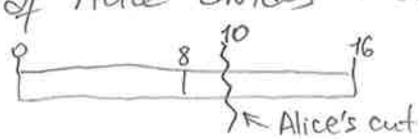


If she values ham as $\frac{2}{3}$ and vegetarian as $\frac{1}{3}$ of the total value, then we see that the total value of her share is $\frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{6}$

So Alice should be very happy as she got $\frac{5}{6}$ of total Value

(b) If Alice divides then her cut should be at

EXPLAIN!!!



since this is the only way to cut into 2 shares of equal value in Alice's value system.

Now the choice is up to Bob, who will take the leftmost half, which is of 100% value in the Bob's value system ■

- Q: What should we do if there are more than 2 players?

Historically: * the Divider-Chooser Method was generalized to the case of 3 players by Polish mathematician Hugo Steinhaus in 1943 and was further generalized to N players by Princeton mathematician Harold Kuhn. in 1967
* a completely different method was proposed in 1964 by a mathematician at Iowa State University, called A. Fink.

These two generalizations are known under the names "The Lone-Divider Method" and "The Lone-Chooser Method"

- The "Lone-Divider Method" for 3 players

Step 0: Choose one of the 3 players as a Divider, while the other two will be the Choosers (use a random draw for that) Call them D, C_1, C_2.

Step 1: The divider D divides the assets into 3 shares of equal value in his value system (he will get one of the shares).

Step 2 (Key Step): Each of C_1 and C_2 write on a slip of paper (or they can declare in some other way) which of the shares are fair to them. After that, both lists are revealed.

Step 3: * If in total these two lists list 2 or 3 shares, then it is always possible to assign C_1 one of the shares on his/her list, to C_2 - one of the shares on his/her list, which is not yet assigned to C_1, while D gets the remaining third share.

* If there is only 1 share call S_1 listed (note: at least 1 should be listed) then we give one of the remaining two shares, called S_2, S_3, to the divider (say D gets S_3). After that we combine S_1 and S_2 and split them b/w C_1 & C_2 via the Divider-Chooser Method (3)

- Let us explain why this is a fair-division method.
  - In the first case, when there are ≥ 2 shares listed by C_1 & C_2, we see that each of C_1 and C_2 gets a fair share (i.e. a share worth of $\geq \frac{1}{3}$ in his/her value system), while the divider gets a share worth of exactly $\frac{1}{3}$ in his/her value system $\Rightarrow$ also fair
  - In the second scenario, D gets s_3 whose value is exactly $\frac{1}{3}$ in D's opinion $\Rightarrow$ D got a fair share.

    However, s_3 was not on the list of C_1 nor C_2 $\Rightarrow$ they value s_3 at less than $\frac{1}{3}$ of total value $\Rightarrow$ s_1+s_2 has a value $>\frac{2}{3}$ of total Value for both of them. Since using the Divider-Chooser Method each gets at least $\frac{1}{2}\cdot$ Value of s_1+s_2, we see that C_1, C_2 got shares worth of $(\geq \frac{1}{2})\cdot(>\frac{2}{3}) > \frac{1}{3}\cdot$ Total Value
- Let us now illustrate this method by the following simple example:

Example: Alice, Bob, and Dylan are dividing a cake using the Lone-Divider Method. Their valuation of shares is depicted by:

	s_1	s_2	s_3
Alice	35%	25%	
Bob	47%		33%
Dylan		$33\frac{1}{3}\%$	$33\frac{1}{3}\%$

(a) Who was the divider?

(b) What is the outcome of the Lone-Divider method applied in this case?

► Step 1: Fill in the empty entries by recalling that the sum in each row should be 100%.

Then: s_3's value for Alice is $100-35-25=40\%$

s_2's value for Bob is $100-47-33=20\%$

s_1's value for Dylan is $100-33\frac{1}{3}-33\frac{1}{3}=33\frac{1}{3}\%$

Step 2: Divider must have all entries equal $\Rightarrow$ Divider is Dylan!

Step 3 (Bidding): Alice's fair shares: s_1, s_3, Bob's fair shares: s_1.

So: Must give s_1 to Bob $\Rightarrow$ s_3 – to Alice $\Rightarrow$ s_2 – to Dylan ■