Lecture #9 09/19/2016

- Last time: - The Divider-Chooser Method (2 players only)
  - The Lone-Divider Method (≥ 3 players)

Before we switch to new methods, let's do two more examples.

Example 1 (Problem 3.3.25) Four players (P_1, P_2, P_3, P_4) divide a cake using the lone-divider method. The divider divides the cake into four slices (s_1, s_2, s_3, s_4). The values of slices to each player are expressed by the following Table:

	s_1	s_2	s_3	s_4
P_1	20%	32%	28%	20%
P_2	25%	25%	25%	25%
P_3	15%	15%	30%	40%
P_4	24%	24%	24%	28%

(a) Who was the divider?

(b) Find a fair division of the cake.

(a) The divider must have the same % for each slice
$\Rightarrow$ $\boxed{P_2 \text{ is the divider}}$ as he has 25% for each slice.

(b) Now we have to write down fair shares (slices) for each other player. Note that a slice is fair if the corresponding player values it at $\geq \frac{100\%}{4} = 25\%$

Fair Shares for P_1: s_2, s_3
for P_3: s_3, s_4
for P_4: s_4.

Since P_4 has only one fair slice, we MUST give s_4 to P_4.
Next, P_3 has 2 fair slices, but as s_4 is already given to P_4, we MUST give s_3 to P_3. Next, P_1 has two fair slices s_2, s_3, but s_3 is already given to P_3 $\Rightarrow$ MUST give to P_1.
The remaining share s_1 is left to P_2.

$\boxed{\text{Answer}}$: P_1 gets s_2, P_2 gets s_1, P_3 gets s_3, P_4 gets s_4 ∎

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Example 2: Recall an example from last time, where Alice and Bob want to split a joint sub into 2 shares by using the divider-chooser method; but today we choose Alice to be a divider.

(a) Where should Alice make the cut?

(b) Which share will Bob choose and what is its value in Bob's value system?

Recall that Alice likes Ham twice more than vegetarian part. Hence she must make a cut somewhere in ham half (since the value of vegie-half is < 50%)

Let us denote by x the length of the totally-ham-part which will be to the right of Alice's cut. Clearly $0 \le x \le 8$. Then the value of that part equals $\frac{x}{16-8} = \frac{x}{8}$ of the total value of ham-half to Alice. Since Alice values the ham part of sub as $\frac{2}{3} \cdot V$ ← total value of sub, we get an equation

$$\boxed{\frac{x}{8} \cdot \frac{2}{3} V = \frac{1}{2} V}$$

(as Alice makes a cut so that both shares are worth of $\frac{1}{2}V$)

So: $\frac{x}{8} \cdot \frac{2}{3} = \frac{1}{2} \Rightarrow x = 8 \cdot \frac{3}{2} \cdot \frac{1}{2} = 6 \Rightarrow x = 6$

(a) Alice cuts 6 inches from the right end.

(b) As Bob values the vegetarian part as 100%, it is clear he will choose the left share, whose value is 100% in Bob's Value System ∎

! Again, we saw it is much better to be a chooser than divider.

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• Some comments on Homework

* As we mentioned, given a table with share's values in %, we MUST always get 100% when we add all numbers in rows. HOWEVER: If the table contains cost of each share in each of the player's opinion (now in \$, not in %), then it is not necessary to have the same sum of numbers in each row. Instead, you may always transfer this table into the previous one (%). For example, if a player values shares s_1, s_2 at 4\$ and s_3 at 8\$ (only 3 shares), then it means that he/she values s_1, s_2 at $\frac{4}{4+4+8} = \frac{1}{4}$ (=25%), while s_3 at $\frac{8}{4+4+8} = \frac{1}{2}$ (=50%).
* As mentioned last time, if you are given a table of value systems with at most one box empty at each row, you can immediately fill it by using the above remark about 100%.

! If there any q-s on hwk: come to O.H. after this lecture.

• The Lone-Chooser Method

As mentioned last time, the divider-chooser method for 2 players admits two different generalizations for the situation of ≥3 players. And as the name suggests, while in the Lone-Divider method we had one divider and many choosers, now we will have an opposite situation.

In the Lone-Chooser Method, one player will play role of a chooser, while all others will be dividers.

First let us explain the Lone-Chooser Method for 3 players.

Step 0: Decide by a random draw who is the chooser (C) and call the other two players - the dividers ($D1$, $D2$).

Step 1: D_1 and D_2 divide S between themselves into 2 fair shares using the divider-chooser method.

Step 2: After that each of D_1 and D_2 divides his share into 3 subshares of equal value in their respective value systems.

Step 3: The chooser C "comes into the play" and picks one of the three subshares of D_1 and one from the 3 subshares of D_2. After that D_1 and D_2 keep to themselves the remaining two subshares.

We can depict that as

$$\begin{tikzcd}[row sep=small] & & S_{1a} \\ & S_1 \arrow[ur] \arrow[r] \arrow[dr] & S_{1b} \\ S \arrow[ur] \arrow[dr] & & S_{1c} \\ & S_2 \arrow[r] \arrow[ur, phantom] \arrow[dr] & S_{2b} \\ & & S_{2c} \end{tikzcd}$$

$$S_2 \to S_{2a}$$

The idea is that C chooses the "best" share out of $\{S_{1a}, S_{1b}, S_{1c}\}$ and $\{S_{2a}, S_{2b}, S_{2c}\}$.

Q: Why is it a fair-division method?

A: Due to the fairness of the divider-chooser method, both s_1 and s_2 are worth of at least $\frac{1}{2}$ Total value to D_1 and D_2, respectively. Since they divided them into 3 subshares of equal value and got 2 of them at the end $\Rightarrow$ they have at least $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$ of the total Value.

In C's value system

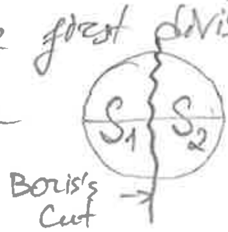
As for the chooser, he got $\geq \frac{1}{3} \cdot (\text{Value } s_1) + \frac{1}{3} \cdot (\text{Value } s_2)$, but $(\text{Value } s_1) + (\text{Value } s_2) = \text{Total value}$. Hence, C also got $\geq \frac{1}{3}$ Total Value

Let us consider an example illustrating this method.

Example 3 (Problem 3.4.35)

Angela, Boris, Carlos are dividing the vanilla-strawberry cake using the lone-chooser method. The following pictures show how each player values each half of the cake.

We assume that all cuts are from the center to the edge of the cake. Suppose that Angela and Boris are dividers and Carlos is the chooser. In the first division Boris cuts the cake vertically through the center



with Angela choosing the left half S_1, Boris left with the right half S_2.

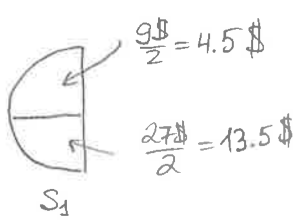
(a) Describe how Angela would subdivide S_1 into three pieces

(b) — // — Boris — // — S_2 — // —

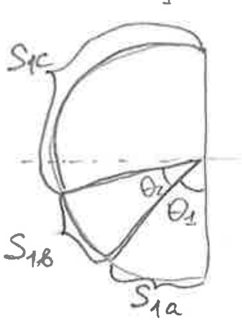
(c) Describe a possible final fair division of the cake.

(d) For the fair division from part (c), find the value (in $) of each share in the eyes of each player.

(a)



The value of S_1 in the eyes of Angela is $4.5 + 13.5 = 18\$$. When dividing into 3 subshares, they must be $\frac{18\$}{3} = 6\$$ each. Since $4.5\$ < 6\$$, both cuts must be made at the strawberry part.



If θ_1 and θ_2 are angles as shown on the picture, then

$$Cost(S_{1a}) = \frac{\theta_1}{90^\circ} \cdot 13.5\$, \quad cost(S_{1B}) = \frac{\theta_1}{90^\circ} \cdot 13.5\$$$

Since both of them must be $6\$$, we find

$$\theta_1 = \theta_2 = \frac{6\$}{13.5\$} \cdot 90^\circ = \frac{4}{9} \cdot 90^\circ = 40^\circ$$

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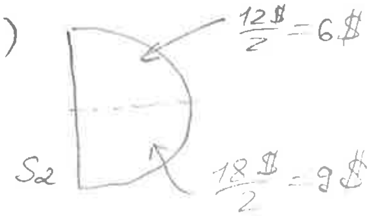
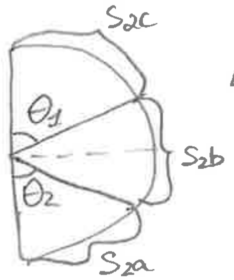
(Continuation)

(b)



The value of S_2 in the eyes of Boris is $6+9=15\$$. Since he divides it into 3 subshares of equal value, each of them must be worth of $\frac{15\$}{3} = 5\$$.
As $6\$ > 5\$$, $9\$ > 5\$$, we see that one of the cuts must be in vanilla part and another - in the strawberry part.



Let θ_1 and θ_2 be the angles on the picture. Then $cost(S_{2a}) = \frac{\theta_2}{90^\circ} \cdot 9\$$, $cost(S_{2c}) = \frac{\theta_1}{90^\circ} \cdot 6\$$.
As each of them must be $5\$$, we find

$$\frac{\theta_2}{90^\circ} \cdot 9\$ = 5\$ \Rightarrow \theta_2 = \frac{5\$}{9\$} \cdot 90^\circ = 50^\circ$$

$$\frac{\theta_1}{90^\circ} \cdot 6\$ = 5\$ \Rightarrow \theta_1 = \frac{5\$}{6\$} \cdot 90^\circ = 75^\circ$$

(c) • Since Carlos likes Vanilla twice more than strawberry, it is clear that he will choose S_{1c} out of $\{S_{1a}, S_{1b}, S_{1c}\}$. Indeed in Carlos eyes: $cost(S_{1a}) = \frac{24\$}{2} + \frac{12\$}{2} \cdot \frac{10^\circ}{90^\circ} = 12\$ + \frac{6}{9}\$ = 12\frac{2}{3}\$$, while the cost of S_{1b}, S_{1a} is clearly less (you might need to compute them!).

• When choosing between $\{S_{2a}, S_{2b}, S_{2c}\}$, their costs in Carlos eyes:

$$cost(S_{2a}) = \frac{50^\circ}{90^\circ} \cdot \frac{12\$}{2} = \frac{30}{9}\$ = \frac{10}{3}\$$$

$$cost(S_{2c}) = \frac{75^\circ}{90^\circ} \cdot \frac{24\$}{2} = \frac{5}{6} \cdot 12\$ = 10\$$$

$$cost(S_{2b}) = \frac{24\$}{2} + \frac{12\$}{2} - cost(S_{2a}) - cost(S_{2c}) = \frac{14}{3}\$$$

$$\boxed{\begin{array}{l} \text{Hence: - Carlos will choose } S_{1c} \text{ and } S_{2c}\text{, whose total cost is } 22\tfrac{2}{3}\$ \\ \quad\quad \text{- Angela will be left with } S_{1a}, S_{1b} \text{ worth of } \tfrac{2}{3}\cdot(4.5+13.5)\$ = 12\$ \\ \quad\quad \text{- Boris will be left with } S_{2a}, S_{2b} \text{ worth of } \tfrac{2}{3}\cdot(6\$+9\$) = 10\$ \end{array}}$$

(d) See above.